Angular Momentum in Quantum Mechanics

Advanced Undergraduate and Early Postgraduate Notes

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Chapter 1

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Chapter Abstract

Angular momentum is often introduced in mechanics as the moment of linear momentum, a vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ associated with rotation about a chosen origin. In quantum mechanics it becomes much more than a convenient mechanical quantity. Angular momentum is one of the basic organizing principles of the theory because it is tied directly to rotational symmetry. Whenever a Hamiltonian is invariant under rotations, angular momentum conservation, degeneracy, and selection rules follow. The familiar degeneracy of magnetic substates, the shapes of atomic orbitals, the splitting of beams in a Stern–Gerlach apparatus, and the algebra behind spin all come from this same structure.

The quantum theory also forces us to revise the classical picture. The three Cartesian components of angular momentum are represented by noncommuting operators, so a state cannot in general possess sharp values of L_x, L_y, and L_z simultaneously. Instead, one usually labels states by the compatible observables L^2 and one component, conventionally L_z. Their eigenvalues are discrete, and the allowed magnetic quantum numbers form finite ladders. Spin enlarges the idea still further: it is angular momentum without being the literal rotation of a spatially extended object. For spin-$\frac{1}{2}$, the Pauli matrices give a complete and experimentally accurate description.

This chapter develops angular momentum from its operator definition through the commutation algebra, eigenvalue spectrum, matrix representations, spin, Clebsch–Gordan coefficients, rotations, central potentials, and the hydrogen atom. The emphasis is mathematical enough to be reliable but physical enough to show why the formalism matters.

1.1 Introduction: Why Angular Momentum Matters

Angular momentum enters classical mechanics as the quantity that measures the rotational state of motion of a particle about a chosen origin. If a particle is at position $\mathbf{r}$ and has linear momentum $\mathbf{p}$, its orbital angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{1.1}$$

The vector $\mathbf{L}$ is perpendicular to the plane spanned by $\mathbf{r}$ and $\mathbf{p}$, with direction fixed by the right-hand rule. Its magnitude is $L = rp \sin \alpha$, where α is the angle between $\mathbf{r}$ and $\mathbf{p}$. In this familiar setting angular momentum is useful because it is conserved by central forces, because it

gives a compact description of rotational motion, and because it is the rotational analogue of linear momentum.

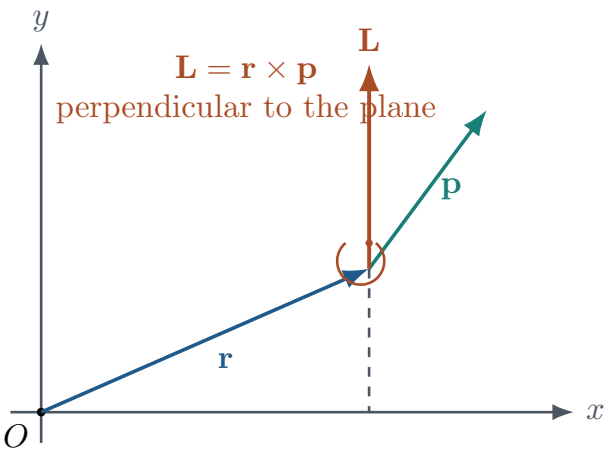

Figure 1.1: Classical orbital angular momentum is the vector product $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. It is perpendicular to the plane containing the position and momentum vectors.

Quantum mechanics keeps the classical idea but changes its status. Position and momentum become operators, and therefore angular momentum becomes an operator. The transition is not merely a matter of placing hats on classical symbols. Since $\widehat{x}$ and $\widehat{p}_x$ do not commute, the components of $\widehat{\mathbf{L}}$ inherit a noncommutative algebra. This algebra has direct physical consequences: one cannot in general assign sharp simultaneous values to L_x, L_y, and L_z. The most that can be done for an ordinary angular momentum is to specify the total magnitude squared L^2 and one component, conventionally L_z. This is the origin of quantum numbers such as l and m.

The importance of angular momentum in quantum mechanics is deeper than the classical analogy suggests. Angular momentum is the generator of rotations. A small rotation of a quantum state is produced by an operator of the form

$$1 - \frac{\mathrm{i}}{\hbar}\,\delta\theta\,\hat{\mathbf{n}} \cdot \mathbf{J},$$

where $\hat{\mathbf{n}}$ is the rotation axis and $\mathbf{J}$ is the angular momentum associated with the system. Thus angular momentum is the bridge between rotational symmetry and observable consequences. If a Hamiltonian is invariant under rotations, then the corresponding angular momentum is conserved. If a system has spherical symmetry, its states can be organized into multiplets labelled by angular momentum quantum numbers. If a transition is caused by an interaction with definite rotational properties, selection rules follow.

This symmetry viewpoint explains why angular momentum appears almost everywhere in quantum mechanics. Atomic spectra are organized by angular momentum because the Coulomb potential is rotationally symmetric. Central potentials lead naturally to spherical harmonics $Y_l^m(\theta, \phi)$, and these functions are eigenfunctions of $\widehat{L}^2$ and $\widehat{L}_z$. Spin systems are described by angular momentum algebras even when there is no literal motion in ordinary space. The rules for adding angular momenta determine the structure of atomic fine structure, hyperfine structure, molecular rotations, nuclear spin states, and many scattering amplitudes.

It is useful from the beginning to distinguish orbital angular momentum from spin angular momentum. Orbital angular momentum is associated with the spatial wavefunction of a particle.

For one particle in three-dimensional space it is represented by

$$\widehat{\mathbf{L}} = \widehat{\mathbf{r}} \times \widehat{\mathbf{p}}.$$

Spin angular momentum, written $\widehat{\mathbf{S}}$, is intrinsic. It is not obtained from $\mathbf{r} \times \mathbf{p}$ and it is not the classical rotation of a tiny rigid body. A spin-$\frac{1}{2}$ electron, for example, has only two independent spin states along any chosen axis, usually called spin up and spin down. Treating the electron as a small charged sphere rotating in space leads to contradictions with relativity and with the observed magnetic moment. Spin is instead an internal quantum degree of freedom whose components obey the same angular-momentum commutation relations.

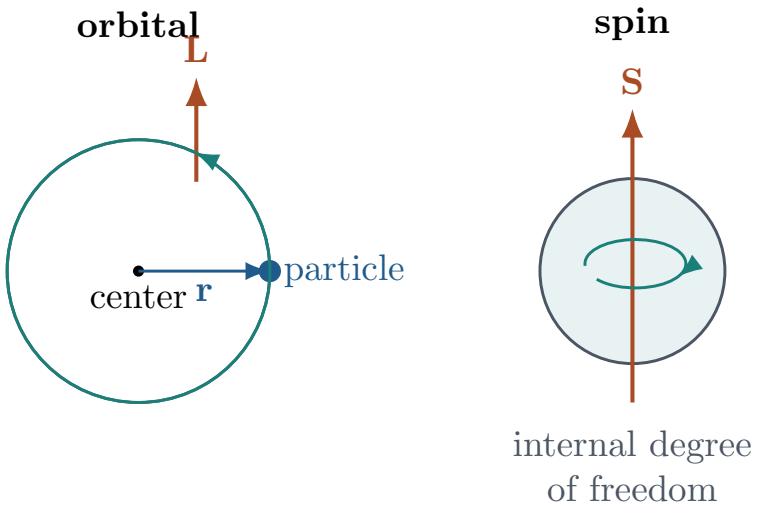

Figure 1.2: Orbital angular momentum is tied to spatial motion about a center. Spin is an intrinsic degree of freedom and should not be interpreted as a literal classical spinning motion of an extended object.

The distinction is not a separation of importance. In real atoms both orbital and spin angular momenta occur and may combine to form total angular momentum. For a single electron in an atom one often writes

$$\widehat{\mathbf{J}} = \widehat{\mathbf{L}} + \widehat{\mathbf{S}}.$$

The total angular momentum $\widehat{\mathbf{J}}$ is the object that controls many rotational properties of the atom. Its quantum numbers determine degeneracies in magnetic fields, allowed optical transitions, and the form of spin-orbit corrections.

There is a subtle but important lesson here. It is tempting to picture a quantum state with angular momentum as a little arrow pointing somewhere in space. This picture can be helpful if used carefully, but it is incomplete. A state $|l, m\rangle$ has a definite value of L^2 and a definite value of L_z, yet it does not have definite values of L_x and L_y. The vector model says that the magnitude of the angular momentum is $\sqrt{l(l+1)}\hbar$, while the projection on the z-axis is $m\hbar$. The remaining transverse direction is not a hidden classical direction waiting to be revealed; rather, it is not specified by the state.

This is why angular momentum is a particularly good place to learn the logic of quantum theory. It contains the central themes of the subject in a compact form: noncommuting observables, compatible sets of operators, discrete spectra, degeneracy, symmetry, conserved quantities, and the relation between abstract Hilbert-space states and measurable results. It also illustrates the power of algebraic methods. Many important results, including the allowed eigenvalues

of angular momentum, follow without solving a differential equation. Ladder operators and commutation relations are enough.

The route through the chapter follows this logic. We begin with the orbital angular momentum operator and its differential representation. We then derive the commutation relations and use them to find eigenvalues and eigenvectors. Matrix representations make the abstract algebra concrete and prepare the way for spin-$\frac{1}{2}$ systems and Pauli matrices. After that we study how angular momenta are added, how Clebsch–Gordan coefficients transform between different bases, and how angular momentum generates rotations. Finally, the formalism is applied to central fields and the hydrogen atom, where the abstract quantum numbers become the familiar labels n, l, m of atomic orbitals.

The unifying idea is this: angular momentum is not only something a quantum system may possess; it is also a language in which rotational symmetry speaks. Once that language is learned, a wide range of physical systems become more transparent.

From a conserved vector to a quantum organizing principle

In classical mechanics angular momentum is first encountered as a conserved vector in problems with no external torque. If a planet moves under an inverse-square gravitational force, its angular momentum is constant; the orbit remains in a plane; and the areal velocity is constant. The same idea appears for a bead moving on a circular hoop, a rigid body rotating about a fixed axis, and a charged particle in a central force field. The common feature is that rotations of the system do not change the physics. A direction may be convenient for calculation, but no direction is built into the law itself.

Quantum mechanics keeps this connection but expresses it in a different grammar. The conserved quantity is now an operator, and the statement of conservation is a commutator equation. If the Hamiltonian has rotational symmetry, then it commutes with the appropriate angular momentum. This is not merely a formal translation. It tells us which operators can be diagonalized together, how energy levels are grouped, and which transitions are forbidden by symmetry. In atomic physics, for example, one can often predict large parts of the spectral pattern before solving the radial equation in detail.

The quantum treatment also separates two ideas that coincide too easily in elementary classical pictures: magnitude and direction. Classically, once a vector $\mathbf{L}$ is specified, all its components are specified. Quantum mechanically a state may have definite L^2 and definite L_z, but it cannot also possess definite L_x and L_y. The phrase "angular momentum vector" must therefore be used carefully. It is still useful because the components transform like a vector under rotations, but it is not a hidden arrow with pre-existing Cartesian components.

This distinction matters in experiments. If a Stern–Gerlach apparatus is aligned along z, it measures the spin projection along z. If the same apparatus is rotated to the x-direction, it measures a different observable. A beam prepared as spin up along z does not simply contain particles whose internal arrows point in the $+z$ direction in the classical sense. When sent through an x-oriented apparatus it splits into two parts, with probabilities determined by the overlap between the z-basis and the x-basis. The experimental fact agrees with the Hilbert-space description, not with a naive classical-vector model.

Angular momentum is also a good example of the way quantum numbers arise. The integer l is not introduced as a label by hand. It appears because the ladder of m-values must terminate if the norm of states is to remain finite and nonnegative. The degeneracy $2l + 1$ is not an accidental counting rule. It is the dimension of the irreducible angular momentum multiplet. This is a recurring pattern in quantum mechanics: the algebra of observables constrains the possible measurement outcomes.

Finally, angular momentum provides a natural bridge from elementary wave mechanics to the language of modern quantum theory. In the hydrogen atom it appears as differential operators and spherical harmonics. In spin systems it appears as finite matrices. In rotations it appears as an exponential generator. In coupled systems it appears through tensor products and Clebsch–Gordan coefficients. These are different representations of one underlying structure. Learning to move between them is one of the main goals of this chapter.

What angular momentum does in actual quantum problems

It is helpful to separate three roles that angular momentum plays in practical calculations. First, it supplies labels for states. Instead of describing an atomic state only by a complicated wavefunction, we label it by quantum numbers such as n, l, m, or by n, l, j, m_j when spin is included. These labels tell us how the state transforms under rotations and what measurement outcomes are possible for angular momentum components. A label such as p-state or d-state is therefore not just spectroscopic vocabulary; it is a statement about an irreducible angular momentum subspace.

Second, angular momentum explains degeneracy. If a Hamiltonian is rotationally invariant, it cannot distinguish states that differ only by orientation. The different m-values in a fixed l multiplet represent different projections along an arbitrarily chosen axis. In the absence of a physical axis, such as an external field, these states must have the same energy. When a field is turned on, it gives the system an axis, and the degeneracy can split. This is why the same quantum number m that looks like a mathematical label in a central potential becomes directly observable in magnetic splitting experiments.

Third, angular momentum restricts transitions. Many matrix elements vanish because the initial state, final state, and interaction operator cannot combine to form a rotational scalar. This is the origin of selection rules. Selection rules are not approximate guesses; when the symmetry assumptions are exact, the corresponding forbidden transitions have exactly zero amplitude. When weak symmetry-breaking effects are present, forbidden transitions may become weakly allowed. Thus angular momentum not only organizes stationary states but also controls how systems change from one state to another.

These three roles are closely related. State labels come from simultaneous eigenstates of commuting operators. Degeneracies come from symmetry and representation dimension. Selection rules come from transformation properties under the same symmetry. Angular momentum is valuable because it lets us reason about all three at once. In many real problems, especially in atomic and nuclear physics, the full Hamiltonian may be complicated, but its rotational properties are simple. Angular momentum theory extracts exact information from that simplicity.

The chapter will repeatedly return to a useful pattern. Identify the symmetry; identify the generator; find a commuting set of operators; label the states; then use the algebra to derive consequences. This pattern is broader than angular momentum, but angular momentum is the cleanest and most physically rich example for learning it.

Why the subject rewards algebraic thinking

Students often first meet quantum mechanics through differential equations: solve the Schrodinger equation, normalize the wavefunction, compute probabilities. Angular momentum shows that this is only one side of the subject. Sometimes the operator algebra gives the answer before any differential equation is solved. The allowed eigenvalues of L^2 and L_z, the number of magnetic substates, and the form of spin-$\frac{1}{2}$ matrices all follow from commutation relations and normalization.

This algebraic approach is not less physical. It is physical in a different way. Differential equations emphasize the shape of the wavefunction in space. Algebra emphasizes the structure imposed by symmetry. In a central potential, both views meet: the algebra predicts the possible angular quantum numbers, and the differential equation supplies the radial functions and energy values. When the two viewpoints agree, the result is especially robust.

Angular momentum also teaches economy. Instead of solving separately for every orientation of a p-orbital, one recognizes a single $l = 1$ multiplet. Instead of treating four two-spin product states as unrelated, one reorganizes them into a triplet and singlet. Instead of computing every transition integral blindly, one first asks which angular momentum couplings are allowed. This economy is one of the reasons angular momentum theory appears so frequently in advanced physics.

The algebraic viewpoint will become increasingly important in later subjects. The same ideas underlie representations of symmetry groups, tensor operators, selection rules, addition of isospin in nuclear physics, polarization states of photons, and qubit rotations in quantum information. A solid understanding of angular momentum therefore pays off well beyond the hydrogen atom.

1.2 Angular Momentum Operators

For a single particle moving in three-dimensional space, the orbital angular momentum operator is obtained from the classical expression $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ by replacing position and momentum with their quantum operators. In the position representation,

$$\widehat{\mathbf{r}} = (x, y, z), \qquad \widehat{\mathbf{p}} = -\mathrm{i}\hbar\nabla. \tag{1.2}$$

Thus

$$\widehat{\mathbf{L}} = \widehat{\mathbf{r}} \times \widehat{\mathbf{p}}. \tag{1.3}$$

The word "orbital" is important. This operator acts on the spatial part of the wavefunction. It says how the wavefunction changes under rotations of the coordinates. Spin angular momentum, studied later, obeys the same algebra but is represented on an internal spin space rather than by the differential operator $\mathbf{r} \times (-\mathrm{i}\hbar\nabla)$.

Writing out the cross product gives the Cartesian components

$$\widehat{L}_x = y\widehat{p}_z - z\widehat{p}_y, \tag{1.4}$$

$$\widehat{L}_y = z\widehat{p}_x - x\widehat{p}_z, \tag{1.5}$$

$$\widehat{L}_z = x\widehat{p}_y - y\widehat{p}_x. \tag{1.6}$$

Using $\widehat{p}_x = -\mathrm{i}\hbar\,\partial/\partial x$, and similarly for y and z, one obtains the differential forms

$$\widehat{L}_x = -\mathrm{i}\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \tag{1.7}$$

$$\widehat{L}_y = -\mathrm{i}\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \tag{1.8}$$

$$\widehat{L}_z = -\mathrm{i}\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \tag{1.9}$$

These operators are first-order differential operators. They do not change the radial coordinate $r = \sqrt{x^2 + y^2 + z^2}$; instead, they differentiate along angular directions. This is why angular momentum becomes especially simple in spherical coordinates.

A particularly useful identity is the expression for $\widehat{L}_z$ in cylindrical or spherical coordinates. Since

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi,$$

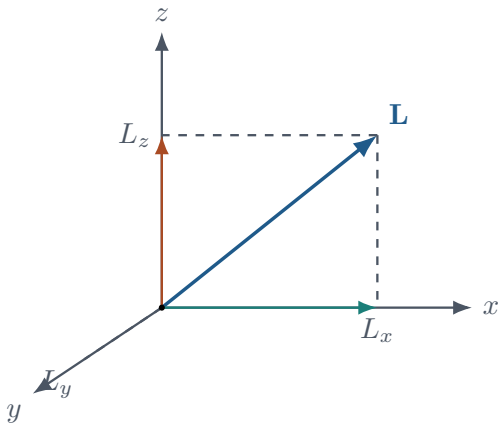

Figure 1.3: The components L_x, L_y, L_z are projections of angular momentum on the coordinate axes. Quantum mechanically the components are observables, but not mutually compatible observables.

one finds

$$\widehat{L}_z = -\mathrm{i}\hbar \frac{\partial}{\partial \phi}. \tag{1.10}$$

This compact result is one reason the z-component is conventionally chosen when labelling angular momentum states. Rotations about the z-axis are generated by changes in the azimuthal angle ϕ, and the eigenfunctions of $\widehat{L}_z$ have the simple dependence $\mathrm{e}^{\mathrm{i}m\phi}$.

The square of the angular momentum operator is

$$\widehat{L}^2 = \widehat{L}_x^2 + \widehat{L}_y^2 + \widehat{L}_z^2. \tag{1.11}$$

In spherical coordinates it takes the form

$$\widehat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \, \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right]. \tag{1.12}$$

This is the angular part of the Laplacian, multiplied by $-\hbar^2$. It contains no derivative with respect to r. Therefore $\widehat{L}^2$ acts only on the angular shape of a wavefunction. When the wavefunction is written as $R(r)Y(\theta,\phi)$, $\widehat{L}^2$ acts on Y and leaves R untouched.

Angular momentum operators are observables, so they must be Hermitian on their proper domains. For example, $\widehat{L}_z = -\mathrm{i}\hbar\,\partial/\partial\phi$ is Hermitian for wavefunctions that are single-valued and periodic in ϕ:

$$\psi(\phi + 2\pi) = \psi(\phi).$$

Indeed,

$$\begin{aligned} \int_0^{2\pi} \psi^* (-\mathrm{i}\hbar \partial_\phi \chi)\, \mathrm{d}\phi &= -\mathrm{i}\hbar \left[\psi^* \chi\right]_0^{2\pi} + \int_0^{2\pi} (\mathrm{i}\hbar \partial_\phi \psi^*) \chi \, \mathrm{d}\phi \\ &= \int_0^{2\pi} \left[(-\mathrm{i}\hbar \partial_\phi \psi)^*\right] \chi \, \mathrm{d}\phi, \end{aligned} \tag{1.13}$$

where the boundary term vanishes because of periodicity. The same idea applies to $\widehat{L}_x$ and $\widehat{L}_y$, though the integration by parts is less compact in Cartesian coordinates. Hermiticity ensures real measurement outcomes and orthogonality of eigenfunctions with distinct eigenvalues.

Measuring L_z means measuring the component of orbital angular momentum along the selected z-axis. The result, if the state is an eigenstate of $\widehat{L}_z$, is a definite number. If the state

is a superposition of L_z eigenstates, the measurement returns one of the allowed eigenvalues with probabilities determined by the expansion coefficients. The same principle applies to L_x or L_y, but because the components do not commute, a state that is sharp in L_z is generally not sharp in L_x.

The z-axis has no intrinsic privilege in an isotropic system. It is chosen because any one component may be diagonalized together with $\widehat{L}^2$, and L_z is algebraically convenient. In an experiment, the choice of component is set by the apparatus: for example, an external magnetic field defines a physical axis. If the field points along z, the energy shift of a magnetic moment often depends on L_z, making m a directly relevant quantum number.

It is also useful to separate three ideas that are sometimes blurred. An operator such as $\widehat{L}_z$ is a rule for acting on states. An eigenvalue such as $m\hbar$ is a number associated with a special state satisfying $\widehat{L}_z|\psi\rangle = m\hbar|\psi\rangle$. A measured value is the number actually obtained in a measurement. If the system is prepared in an eigenstate, the measured value is predictable. If not, the theory gives probabilities.

The dimensions of angular momentum are those of action. Since $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, the unit is length times momentum, or $\mathrm{kg\,m^2\,s^{-1}}$. Quantum mechanically the natural unit is $\hbar$. Thus angular momentum eigenvalues appear as multiples of $\hbar$, and squared angular momenta appear as multiples of $\hbar^2$. The dimensionless numbers l, m, s, m_s, j, m_j carry the discrete structure, while $\hbar$ carries the unit.

Definition: Orbital angular momentum

For a spinless particle in three-dimensional space, the orbital angular momentum operator is

$$\widehat{\mathbf{L}} = \widehat{\mathbf{r}} \times \widehat{\mathbf{p}} = -\mathrm{i}\hbar\, \mathbf{r} \times \nabla.$$

It acts on the spatial wavefunction and generates rotations of the spatial coordinates.

The rest of the chapter can now be viewed as an exploration of the consequences of this definition. The cross product structure leads to a noncommutative algebra. That algebra leads to quantized eigenvalues. The same algebra also describes spin and total angular momentum, even when the concrete representation is no longer the differential operator in Eq. (1.3).

Angular derivatives and rotations of wavefunctions

The differential form of $\mathbf{L}$ has an important geometric meaning. It differentiates a wavefunction along directions tangent to spheres centered at the origin. A radial wavefunction $\psi(r)$ is unchanged by rotations, so all three components of orbital angular momentum annihilate it:

$$L_x\psi(r) = L_y\psi(r) = L_z\psi(r) = 0.$$

This is the operator statement that a spherically symmetric state has $l = 0$. By contrast, a wavefunction with angular dependence changes under rotations and may carry nonzero orbital angular momentum.

The simplest example is an azimuthal phase,

$$\psi(\phi) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}m\phi}.$$

Using $L_z = -\mathrm{i}\hbar\partial/\partial\phi$,

$$L_z\psi = m\hbar\psi.$$

Single-valuedness requires

$$\psi(\phi + 2\pi) = \psi(\phi),$$

so $e^{2\pi i m} = 1$, and therefore m must be an integer for an ordinary orbital wavefunction. This simple argument already hints why orbital angular momentum has integer magnetic quantum numbers.

The other components are less simple in spherical coordinates, but their forms are useful:

$$L_x = i\hbar \left(\sin\phi \, \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \, \frac{\partial}{\partial\phi} \right), \tag{1.14}$$

$$L_y = i\hbar \left(-\cos\phi \, \frac{\partial}{\partial\theta} + \cot\theta \sin\phi \, \frac{\partial}{\partial\phi} \right), \tag{1.15}$$

$$L_z = -i\hbar \frac{\partial}{\partial\phi}. \tag{1.16}$$

These expressions make clear why L_z is normally chosen as the component to diagonalize. Its eigenvalue equation is first order in a single angular coordinate, while L_x and L_y mix θ and ϕ.

The ladder operators also have compact differential forms:

$$L_\pm = \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial\theta} + i \cot\theta \, \frac{\partial}{\partial\phi} \right). \tag{1.17}$$

When acting on spherical harmonics, these operators change m while preserving l. In wave mechanics this action is visible as a change in the azimuthal phase factor and in the polar-angle function.

Domains, boundary conditions, and observability

Hermiticity of angular momentum operators is not only a matter of algebraic appearance. Differential operators require domains: one must specify the class of wavefunctions on which the operator acts. For L_z, periodicity in ϕ removes the boundary term in integration by parts. If a candidate wavefunction is not single-valued, it is not an acceptable orbital wavefunction in ordinary three-dimensional quantum mechanics. In more advanced settings, such as charged particles in topologically nontrivial gauge fields, boundary conditions can become subtler, but for the standard problems of this chapter single-valuedness is the guiding requirement.

The distinction between L_z and L^2 is also operational. Measuring L_z asks for a projection on a chosen axis. Measuring L^2 asks for the total squared angular momentum, which is independent of the orientation of the coordinate system. In an atomic context, L^2 is associated with the type of orbital subspace, such as $s, p, d, \ldots$, while L_z distinguishes the magnetic substates within that subspace. When no external field selects an axis, the different m values are physically equivalent orientations of the same angular momentum multiplet.

One should also avoid identifying the operator with a particular result. The operator L_z contains all possible outcomes and their eigenstates. The number $m\hbar$ is an eigenvalue. A single measurement yields one eigenvalue, not the operator itself. This careful language becomes indispensable when discussing spin measurements, where the same two-dimensional state can be expanded in many different component bases.

Operator construction and the correspondence principle

The expression $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is often motivated by the correspondence principle: classical observables become quantum operators. But the correspondence principle must be used with judgment, because quantum operators may fail to commute. For orbital angular momentum the definition is unambiguous because each component contains products of different coordinates and momenta, such as yp_z or zp_y. Since y commutes with p_z, no ordering ambiguity occurs in L_x. The same is

true for the other components. This is fortunate: the classical cross product transfers cleanly into the quantum theory.

Other classical expressions are not always so simple. A product like xp_x would require a decision about whether the Hermitian operator should be xp_x, $p_x x$, or a symmetrized combination. Angular momentum avoids this issue in Cartesian coordinates because the cross product pairs each coordinate with a different momentum component. This is one reason the operator definition is especially elegant.

The operator $\mathbf{L}$ also illustrates a common principle: generators are differential operators. Linear momentum generates translations because $-\mathrm{i}\hbar\nabla$ differentiates the wavefunction with respect to position. Angular momentum generates rotations because $-\mathrm{i}\hbar\,\mathbf{r}\times\nabla$ differentiates the wavefunction along angular directions. In both cases, the observable is not merely a number assigned to motion; it is the operator that implements a continuous transformation.

The units reinforce the interpretation. A generator must appear in an exponential with a dimensionless argument. For translations,

$$T(\mathbf{a}) = \exp(-\mathrm{i}\mathbf{a}\cdot\mathbf{p}/\hbar),$$

so $\mathbf{a}\cdot\mathbf{p}$ has units of action. For rotations,

$$R(\hat{\mathbf{n}},\theta) = \exp(-\mathrm{i}\theta\,\hat{\mathbf{n}}\cdot\mathbf{L}/\hbar),$$

and the angle θ is dimensionless, so $\mathbf{L}$ again has units of action. The appearance of $\hbar$ is therefore not ornamental; it converts the generator into a phase.

What it means to choose an axis

In abstract discussions one often says, "choose L_z." This does not mean the z-axis is physically preferred. It means that a complete commuting set requires one component, and convention selects the third coordinate axis. In an actual experiment the axis is chosen by the apparatus. A magnetic field, a detector orientation, or a molecular axis can make a direction physically meaningful. If the apparatus is rotated, the measured operator changes.

This point helps avoid a common confusion. A state labelled by m is labelled relative to the chosen quantization axis. If one chooses a different axis, the same physical state may be a superposition of the new component eigenstates. This is not a contradiction. It is the same situation as writing a vector in different bases, except that in quantum mechanics the change of basis affects probability amplitudes for incompatible measurements.

1.3 Commutation Relations of Angular Momentum Operators

The algebra of angular momentum follows from the canonical commutation relations of position and momentum,

$$[x_i, x_j] = 0, \qquad [p_i, p_j] = 0, \qquad [x_i, p_j] = \mathrm{i}\hbar\,\delta_{ij}. \tag{1.18}$$

Here $i, j \in \{x, y, z\}$, and δ_{ij} is the Kronecker delta. These relations say that different coordinates commute, different momentum components commute, but a coordinate and its conjugate momentum do not commute. Since angular momentum is built from products of coordinates and momenta, its components inherit a nontrivial commutation structure.

We derive one representative relation explicitly. Using

$$\widehat{L}_x = yp_z - zp_y, \qquad \widehat{L}_y = zp_x - xp_z,$$

we compute

$$\begin{aligned}[L_x, L_y] &= [yp_z - zp_y, zp_x - xp_z] \\ &= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]\,. \end{aligned} \tag{1.19}$$

The middle two terms vanish. For instance, y, x, and p_z commute with one another except for no conjugate pair, so $[yp_z, xp_z] = 0$. Similarly $[zp_y, zp_x] = 0$. The first term is

$$[yp_z, zp_x] = yp_z zp_x - zp_x yp_z. \tag{1.20}$$

Since $p_z z = zp_z - \mathrm{i}\hbar$, and p_x commutes with y, p_z, z, this becomes

$$\begin{aligned} yp_z zp_x - zp_x yp_z &= y(zp_z - \mathrm{i}\hbar)p_x - zyp_x p_z \\ &= yzp_z p_x - \mathrm{i}\hbar\, yp_x - yzp_x p_z \\ &= -\mathrm{i}\hbar\, yp_x. \end{aligned} \tag{1.21}$$

The last term in Eq. (1.19) is

$$[zp_y, xp_z] = zp_y xp_z - xp_z zp_y. \tag{1.22}$$

The first product is $zxp_y p_z$, because p_y commutes with x. For the second product, $p_z z = zp_z - \mathrm{i}\hbar$, so

$$xp_z zp_y = x(zp_z - \mathrm{i}\hbar)p_y = xzp_z p_y - \mathrm{i}\hbar\, xp_y. \tag{1.23}$$

Therefore

$$[zp_y, xp_z] = zxp_y p_z - xzp_z p_y + \mathrm{i}\hbar\, xp_y = \mathrm{i}\hbar\, xp_y. \tag{1.24}$$

Combining the nonzero terms,

$$\begin{aligned}[L_x, L_y] &= -\mathrm{i}\hbar\, yp_x + \mathrm{i}\hbar\, xp_y \\ &= \mathrm{i}\hbar(xp_y - yp_x) \\ &= \mathrm{i}\hbar L_z. \end{aligned} \tag{1.25}$$

Thus

$$\boxed{[L_x, L_y] = \mathrm{i}\hbar L_z.} \tag{1.26}$$

By cyclic permutation of x, y, z, the other two relations are

$$[L_y, L_z] = \mathrm{i}\hbar L_x, \qquad [L_z, L_x] = \mathrm{i}\hbar L_y. \tag{1.27}$$

In compact notation,

$$[L_i, L_j] = \mathrm{i}\hbar\, \epsilon_{ijk} L_k, \tag{1.28}$$

where ϵ_{ijk} is the Levi-Civita symbol and repeated indices are summed. This is the angular momentum algebra. It is also the Lie algebra of the rotation group in three dimensions, usually denoted $\mathfrak{so}(3)$. The physical meaning is that infinitesimal rotations about different axes do not commute: rotating first about x and then about y is not the same as doing the rotations in the opposite order.

The square $\widehat{L}^2 = L_x^2 + L_y^2 + L_z^2$ has a special role. It commutes with each component. Let us verify this for L_z. Since $\left[L_z^2, L_z\right] = 0$, we need

$$\left[L_x^2, L_z\right] + \left[L_y^2, L_z\right].$$

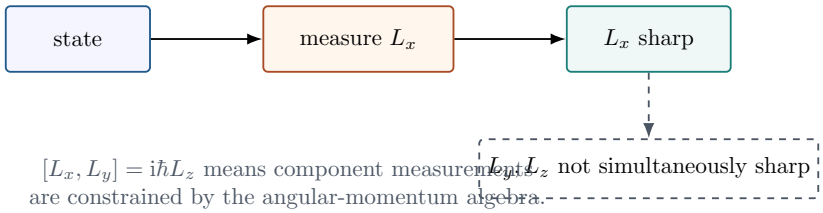

Figure 1.4: Noncommuting angular momentum components cannot be assigned sharp simultaneous values. A measurement that prepares a sharp L_x value generally destroys prior sharpness of L_y or L_z.

Using the identity

$$\left[A^2, B\right] = A\,[A, B] + [A, B]\,A, \tag{1.29}$$

we get

$$\begin{aligned}\left[L_x^2, L_z\right] &= L_x\,[L_x, L_z] + [L_x, L_z]\,L_x = L_x(-\mathrm{i}\hbar L_y) + (-\mathrm{i}\hbar L_y)L_x \\ &= -\mathrm{i}\hbar(L_xL_y + L_yL_x),\end{aligned} \tag{1.30}$$

and

$$\begin{aligned}\left[L_y^2, L_z\right] &= L_y\,[L_y, L_z] + [L_y, L_z]\,L_y = L_y(\mathrm{i}\hbar L_x) + (\mathrm{i}\hbar L_x)L_y \\ &= \mathrm{i}\hbar(L_yL_x + L_xL_y).\end{aligned} \tag{1.31}$$

The two expressions cancel. Hence

$$\left[L^2, L_z\right] = 0. \tag{1.32}$$

The same cyclic reasoning gives

$$\boxed{\left[L^2, L_x\right] = \left[L^2, L_y\right] = \left[L^2, L_z\right] = 0.} \tag{1.33}$$

This result is central to the structure of angular momentum states. Operators that commute can have simultaneous eigenstates, at least when the relevant spectral and domain conditions are satisfied. Thus it is possible to find states that have definite L^2 and definite L_z. We label them by two quantum numbers:

$$|l, m\rangle\,.$$

The first quantum number is associated with the total squared angular momentum, and the second with the selected component. There is no corresponding basis in which L_x, L_y, and L_z are all diagonal, because the components do not commute with one another.

The uncertainty principle makes this statement quantitative. For any two observables A and B,

$$\Delta A\,\Delta B \geq \frac{1}{2}\,|\langle[A, B]\rangle|\,.$$

Taking $A = L_x$ and $B = L_y$ gives

$$\Delta L_x\,\Delta L_y \geq \frac{\hbar}{2}\,|\langle L_z\rangle|\,. \tag{1.34}$$

If $\langle L_z\rangle \neq 0$, the x and y components cannot both be arbitrarily sharp. This is not a defect in experimental technique. It is a structural property of angular momentum in quantum mechanics.

The algebra in Eq. (1.28) is not restricted to orbital angular momentum. Any triple of Hermitian operators J_x, J_y, J_z satisfying

$$[J_i, J_j] = \mathrm{i}\hbar\,\epsilon_{ijk}J_k \tag{1.35}$$

is an angular momentum. Orbital angular momentum is one representation of this algebra; spin is another. This abstraction is powerful. It allows us to derive eigenvalues, ladder rules, and addition rules once, and then apply them to orbital motion, spin, total atomic angular momentum, and many other systems.

The noncommutativity also clarifies a common misconception. A state with definite L_z is not a state in which the angular momentum vector simply points along z. If it were, L_x and L_y would both be zero with certainty. But the algebra forbids such a simultaneous assignment except in the trivial $l = 0$ case. Instead, the state has a definite projection along z, a definite total squared magnitude, and unavoidable transverse uncertainty.

Useful commutator identities

Angular momentum calculations often become manageable once a few operator identities are used consistently. The most important is

$$[AB, C] = A\,[B, C] + [A, C]\,B.$$

It follows directly from adding and subtracting ACB:

$$ABC - CAB = ABC - ACB + ACB - CAB.$$

Similarly,

$$[A, BC] = [A, B]\,C + B\,[A, C]\,.$$

These identities are the operator analogues of product rules. They are especially useful because angular momentum components are products of coordinates and momenta.

Another useful idea is that commuting with L_z measures how an operator changes the magnetic quantum number. If an operator A satisfies

$$[L_z, A] = q\hbar A,$$

then $A\,|l, m\rangle$, if nonzero, has magnetic quantum number $m + q$. The ladder operators are the case $q = \pm 1$. This observation generalizes to spherical tensor operators, where the index q labels how a tensor component changes m. Selection rules in atomic physics are built on this logic.

The Casimir operator

The operator L^2 is sometimes called the Casimir operator of the angular momentum algebra. A Casimir operator commutes with every generator of the algebra. In practical terms, L^2 labels the multiplet, while L_z labels the state inside the multiplet. The same relationship appears for any angular momentum $\mathbf{J}$:

$$J^2\,|j, m\rangle = \hbar^2 j(j+1)\,|j, m\rangle\,, \qquad J_z\,|j, m\rangle = m\hbar\,|j, m\rangle\,.$$

The algebra fixes the structure of each multiplet once j is known.

This is why two systems that look physically different can share the same angular momentum mathematics. A rigid rotor with $l = 1$, a spin-one particle, and the triplet subspace of two spin-$\frac{1}{2}$ particles all have three-dimensional angular momentum representations. The physical Hilbert spaces and Hamiltonians may differ, but the matrices representing J_x, J_y, J_z within the $j = 1$ multiplet obey the same commutation relations.

Noncommuting rotations

The commutator $[L_x, L_y] = \mathrm{i}\hbar L_z$ also has a direct geometric interpretation. Consider two small rotations: one by angle ϵ about x, and another by ϵ about y. In ordinary three-dimensional space, performing them in opposite orders differs by a small rotation about z of order ϵ^2. Quantum mechanically this appears through the product of infinitesimal rotation operators. The lowest-order difference is controlled by the commutator of the generators. Thus the angular momentum algebra is not an arbitrary quantum rule; it is the infinitesimal algebra of rotations.

The factor of $\hbar$ sets the units. If one defines dimensionless generators $K_i = L_i/\hbar$, then

$$[K_i, K_j] = \mathrm{i}\epsilon_{ijk} K_k.$$

Mathematically this is the Lie algebra. Physically the operators L_i carry angular momentum units because their eigenvalues are measured in units of $\hbar$.

How to read a commutator physically

A commutator is sometimes introduced as a formal algebraic object, but it has several physical meanings. The first is compatibility of measurements. If $[A, B] = 0$, the observables A and B can often be diagonalized together, and one may prepare states with sharp values of both. If $[A, B] \neq 0$, sharpness of one observable generally limits sharpness of the other. For angular momentum this is the statement that one component can be sharp together with L^2, but not together with the other two components.

The second meaning is transformation. If G generates a continuous transformation, then $[G, A]$ tells how the observable A changes under that transformation. For example, since L_z generates rotations about z, the commutator $[L_z, L_x] = \mathrm{i}\hbar L_y$ says that the x-component rotates into the y-component. This is exactly what an ordinary vector should do under rotations.

The third meaning is dynamics. In the Heisenberg picture,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}\,[H, A]$$

for an operator with no explicit time dependence. If A commutes with H, it is conserved. Thus the same mathematical object tells us about measurement compatibility, symmetry transformations, and time evolution. Angular momentum is central because all three meanings occur naturally in rotationally invariant systems.

A warning about component uncertainty

The uncertainty relation

$$\Delta L_x\, \Delta L_y \geq \frac{\hbar}{2} |\langle L_z \rangle|$$

does not imply that if $\langle L_z \rangle = 0$, then L_x and L_y can always be simultaneously sharp. The right-hand side is a lower bound, and it may be zero even when other constraints remain. For example, in many states with $\langle L_z \rangle = 0$, the transverse components are still not simultaneously definite because the operators themselves do not commute. The uncertainty relation is a necessary constraint, not a complete classification of possible states.

The stronger statement comes from the absence of common eigenstates. If a state were an eigenstate of both L_x and L_y, then applying the commutator would imply that it has special restrictions involving L_z. For a nonzero angular momentum multiplet, such simultaneous eigenstates do not exist for all three components. The algebra prevents a classical assignment of a full vector direction.

Connection with classical Poisson brackets

The angular momentum commutator has a close classical analogue. In Hamiltonian mechanics, observables are functions on phase space and their infinitesimal transformations are described by Poisson brackets. The classical angular momentum components satisfy

$$\{L_i, L_j\}_{\mathrm{PB}} = \epsilon_{ijk} L_k.$$

Quantization replaces Poisson brackets, roughly, by commutators divided by $\mathrm{i}\hbar$:

$$\{A, B\}_{\mathrm{PB}} \quad \longrightarrow \quad \frac{1}{\mathrm{i}\hbar}\,[A, B]\,.$$

The quantum relation

$$[L_i, L_j] = \mathrm{i}\hbar \epsilon_{ijk} L_k$$

therefore has exactly the expected classical limit. What changes is not the rotational structure but the interpretation of observables. In classical mechanics the components are simultaneously defined functions on phase space. In quantum mechanics they are noncommuting operators, so the same algebra becomes a statement about incompatible measurements.

This comparison is helpful because it shows that noncommutativity is not arbitrary. Rotations already have a non-Abelian structure classically: rotations about different axes do not commute. Quantum mechanics represents that structure through operator commutators. The new feature is that the generators are observables, and their noncommutation affects measurement outcomes.

Commutators and complete sets of commuting observables

A central task in quantum mechanics is to choose a complete set of commuting observables. For a spinless particle in a central potential, a standard choice is

$$H, \quad L^2, \quad L_z.$$

These operators commute with one another, so stationary states can be labelled by their eigenvalues. The set is complete for the angular variables, with the radial quantum number supplied by the radial equation. If spin is included, one may need S^2, J^2, and J_z, depending on the Hamiltonian.

The word "complete" means that the labels distinguish the states within the space under consideration, up to accidental degeneracies or additional symmetries. It does not mean every observable is sharp. In the state $|l, m\rangle$, L_x and L_y are not sharp even though L^2 and L_z are. A complete commuting set is therefore a maximal set of compatible questions, not a complete classical specification of all possible quantities.

This is an important conceptual shift. Classical mechanics often describes a state by enough variables to determine all observables. Quantum mechanics describes a state as a vector in Hilbert space, and different observables correspond to different bases. Angular momentum is one of the clearest examples because the algebra itself tells us which questions can be answered simultaneously.

1.4 Eigenvalues and Eigenvectors of Angular Momentum

The commutation relations make it possible to determine the allowed eigenvalues of angular momentum without solving a differential equation. We work with a general angular momentum $\mathbf{J}$, because the argument applies equally to orbital angular momentum $\mathbf{L}$, spin $\mathbf{S}$, and total

angular momentum **J**. In this section the notation **L** is used for the orbital case, but the algebraic steps should be read as general.

Since $\widehat{L}^2$ commutes with $\widehat{L}_z$, we choose simultaneous eigenstates:

$$\widehat{L}^2 \,|l,m\rangle = \hbar^2 l(l+1)\,|l,m\rangle\,, \tag{1.36}$$

$$\widehat{L}_z \,|l,m\rangle = m\hbar\,|l,m\rangle\,. \tag{1.37}$$

The notation $l(l+1)$ is not an assumption about the answer; it is a conventional and convenient parametrization of the nonnegative eigenvalue of $L^2/\hbar^2$. The algebra will show that l is a nonnegative integer for orbital angular momentum and that m runs from $-l$ to $+l$ in integer steps.

Define the ladder operators

$$\widehat{L}_+ = \widehat{L}_x + \mathrm{i}\widehat{L}_y, \qquad \widehat{L}_- = \widehat{L}_x - \mathrm{i}\widehat{L}_y. \tag{1.38}$$

They are adjoints of one another: $L_+^\dagger = L_-$. They are called raising and lowering operators because they change the m quantum number. The needed commutators follow directly from the angular momentum algebra:

$$[L_z, L_+] = [L_z, L_x] + \mathrm{i}\,[L_z, L_y] = \mathrm{i}\hbar L_y + \mathrm{i}(-\mathrm{i}\hbar L_x) = \hbar L_+, \tag{1.39}$$

$$[L_z, L_-] = [L_z, L_x] - \mathrm{i}\,[L_z, L_y] = \mathrm{i}\hbar L_y - \mathrm{i}(-\mathrm{i}\hbar L_x) = -\hbar L_-. \tag{1.40}$$

Also,

$$\big[L^2, L_\pm\big] = 0. \tag{1.41}$$

Now apply L_+ to $|l,m\rangle$. Because L^2 commutes with L_+,

$$L^2(L_+\,|l,m\rangle) = L_+(L^2\,|l,m\rangle) = \hbar^2 l(l+1)(L_+\,|l,m\rangle).$$

Thus $L_+\,|l,m\rangle$, if nonzero, has the same l. For L_z,

$$\begin{aligned} L_z(L_+\,|l,m\rangle) &= (L_+L_z + [L_z, L_+])\,|l,m\rangle \\ &= L_+(m\hbar\,|l,m\rangle) + \hbar L_+\,|l,m\rangle \\ &= (m+1)\hbar\, L_+\,|l,m\rangle\,. \end{aligned} \tag{1.42}$$

Therefore $L_+\,|l,m\rangle$ is proportional to $|l,m+1\rangle$. Similarly $L_-\,|l,m\rangle$ is proportional to $|l,m-1\rangle$.

To find the proportionality constants, use

$$\begin{aligned} L_-L_+ &= (L_x - \mathrm{i}L_y)(L_x + \mathrm{i}L_y) = L_x^2 + L_y^2 + \mathrm{i}\,[L_x, L_y] \\ &= L_x^2 + L_y^2 - \hbar L_z = L^2 - L_z^2 - \hbar L_z, \end{aligned} \tag{1.43}$$

and

$$L_+L_- = L^2 - L_z^2 + \hbar L_z. \tag{1.44}$$

If

$$L_+\,|l,m\rangle = C_+(l,m)\,|l,m+1\rangle\,,$$

then, assuming normalized states,

$$\begin{aligned} |C_+(l,m)|^2 &= \langle l,m|\,L_-L_+\,|l,m\rangle \\ &= \hbar^2\,\big[l(l+1) - m^2 - m\big] \\ &= \hbar^2\,[l(l+1) - m(m+1)]\,. \end{aligned} \tag{1.45}$$

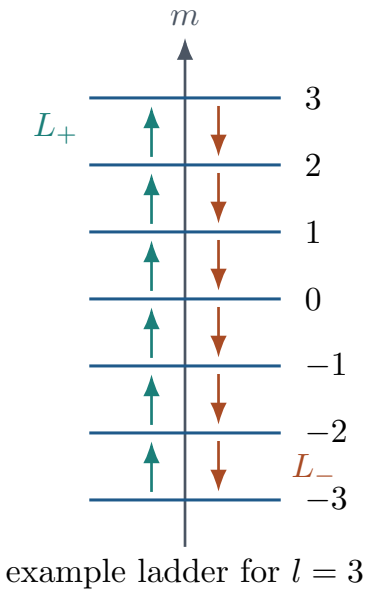

Figure 1.5: For fixed l, the ladder operators move through the allowed m values in unit steps. The ladder terminates at $m = +l$ and $m = -l$.

Choosing the conventional positive real phase,

$$\boxed{L_+ |l, m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\ |l, m+1\rangle\,.} \tag{1.46}$$

Similarly,

$$\boxed{L_- |l, m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\ |l, m-1\rangle\,.} \tag{1.47}$$

These are among the most useful formulas in angular momentum theory.

The ladder cannot continue indefinitely. For a fixed l, all states in the ladder have the same value of L^2, and the expectation value of $L_x^2 + L_y^2$ must be nonnegative:

$$\begin{aligned}\langle l, m| \left(L_x^2 + L_y^2\right) |l, m\rangle &= \langle l, m| \left(L^2 - L_z^2\right) |l, m\rangle \\ &= \hbar^2 \left[l(l+1) - m^2\right] \geq 0.\end{aligned} \tag{1.48}$$

Thus $m^2 \leq l(l+1)$. There must be a highest state $|l, m_{\max}\rangle$ annihilated by L_+:

$$L_+ |l, m_{\max}\rangle = 0.$$

Using Eq. (1.46),

$$l(l+1) - m_{\max}(m_{\max} + 1) = 0.$$

The acceptable root is $m_{\max} = l$. Similarly the lowest state satisfies $m_{\min} = -l$. Since ladder operations change m by one unit, the number of steps from $-l$ to $+l$ is $2l$, so $2l$ must be an integer. For a general angular momentum, l may therefore be integer or half-integer. For orbital angular momentum, single-valuedness of the spatial wavefunction restricts l to integers:

$$l = 0, 1, 2, \ldots, \qquad m = -l, -l+1, \ldots, l-1, l. \tag{1.49}$$

For spin, half-integer values such as $s = \frac{1}{2}$ are allowed.

The number of allowed m values for a fixed l is

$$(l - (-l)) + 1 = 2l + 1. \tag{1.50}$$

This number appears constantly. In a rotationally invariant Hamiltonian, energy often depends on l but not on m, giving a $(2l + 1)$-fold degeneracy. When a magnetic field is applied, the m states may split because the field selects a direction in space.

The magnitude of angular momentum in a state $|l, m\rangle$ is not $l\hbar$. Since the eigenvalue of L^2 is $\hbar^2 l(l + 1)$, the magnitude associated with the vector model is

$$|\mathbf{L}| = \sqrt{l(l + 1)}\,\hbar. \tag{1.51}$$

This is larger than $l\hbar$. The projection on the z-axis is $m\hbar$, whose largest possible value is $l\hbar$. Thus even in the state with maximum projection $m = l$, the angular momentum is not perfectly aligned with the z-axis. There remains transverse angular momentum:

$$L_x^2 + L_y^2 = L^2 - L_z^2.$$

This is a direct consequence of noncommutativity.

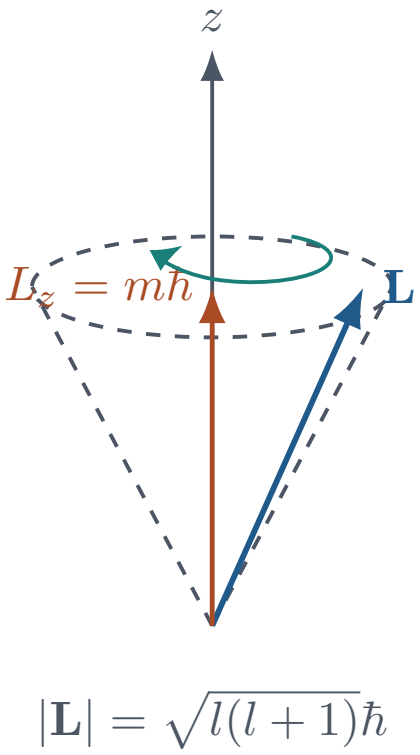

Figure 1.6: The vector model represents a state with definite L^2 and L_z. The length is $\sqrt{l(l + 1)}\hbar$, while the z-projection is $m\hbar$. The transverse direction is not definite.

This picture is sometimes called space quantization. The phrase means that the projection of angular momentum along a chosen axis is quantized. It does not mean that space itself is made of discrete directions. The axis is selected by the experimental arrangement or by our choice of basis, and only the allowed measurement results are discrete. In a spherically symmetric problem, all axes are equivalent, but once a basis is chosen we use m to label the possible projections.

The eigenstates $|l, m\rangle$ form an orthonormal basis within a fixed l subspace:

$$\langle l, m\rangle\, l, m' = \delta_{mm'}. \tag{1.52}$$

For orbital angular momentum these states correspond in the position representation to spherical harmonics:

$$\langle \theta, \phi\rangle\, l, m = Y_l^m(\theta, \phi).$$

The ladder operator method therefore tells us the possible eigenvalues of spherical harmonics before we ever write their explicit functional form. This is one of the major advantages of the algebraic approach.

Angular momentum spectrum

For orbital angular momentum,

$$L^2\,|l, m\rangle = \hbar^2 l(l+1)\,|l, m\rangle\,, \qquad L_z\,|l, m\rangle = m\hbar\,|l, m\rangle\,,$$

with

$$l = 0, 1, 2, \ldots, \qquad m = -l, -l+1, \ldots, l.$$

For a general angular momentum such as spin, l is replaced by j or s, and integer as well as half-integer values may occur.

Why the ladder argument is so powerful

The ladder derivation is a model example of algebraic quantum mechanics. No coordinate system was used, and no wavefunction was solved for. The only ingredients were the commutation relations, Hermiticity, and the positivity of norms. Positivity is essential: the quantity

$$\|L_+\,|l, m\rangle\,\|^2 = \langle l, m|\, L_- L_+\,|l, m\rangle$$

cannot be negative. This requirement forces the ladder to have endpoints. Without endpoints, one could keep raising or lowering m forever while L^2 stayed fixed, eventually violating the bound $m^2 \le l(l+1)$.

The same reasoning explains the square-root factors. They are not arbitrary normalization conventions. They are the norms of the unnormalized raised or lowered states. A common error is to remember only that L_+ raises m and L_- lowers m, while forgetting that the coefficient depends on both l and m. The coefficient becomes zero exactly at the top or bottom of the ladder, which is how the algebra knows when to stop.

Integer versus half-integer angular momentum

The algebra alone allows $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Orbital angular momentum is restricted to integer l because the wavefunction must be single-valued under a 2π rotation in ordinary space. Spin is not represented by a scalar spatial wavefunction, so half-integer values are possible. A spinor may change sign under a 2π rotation without changing observable probabilities, because an overall phase does not affect expectation values. This is one of the clearest signs that spin is not merely hidden orbital motion.

For orbital angular momentum, the functions $Y_l^m(\theta, \phi)$ also enforce the integer rule. The ϕ-dependence is $\mathrm{e}^{\mathrm{i}m\phi}$, so m is an integer. Since m runs from $-l$ to $+l$ in integer steps, l must be an integer as well. In contrast, a spin-$\frac{1}{2}$ state has no ordinary angular coordinate ϕ in physical space. Its two-component spinor structure is a representation of rotations in an internal Hilbert space.

The vector model and its limits

The vector model is helpful because it gives a visual summary of the eigenvalues:

$$|\mathbf{L}| = \sqrt{l(l+1)}\hbar, \qquad L_z = m\hbar.$$

One imagines a vector of fixed length whose projection on the z-axis is fixed, while its transverse direction is undetermined. This image correctly suggests why m is bounded and why the number of projections is finite.

However, the image should not be pushed too far. There is no underlying classical vector with a definite azimuthal angle that we simply do not know. If such an angle existed as an ordinary hidden variable, it would assign definite values to L_x and L_y, contradicting the noncommuting operator structure. The precession cone is a mnemonic for expectation values and uncertainties, not a literal microscopic orbit of the angular momentum vector.

Degeneracy and multiplets

For each l, the states $|l, m\rangle$ form a multiplet:

$$|l, l\rangle, |l, l-1\rangle, \ldots, |l, -l\rangle.$$

The operators $L_\pm$ move within the multiplet, while L^2 has the same eigenvalue for every member. If a Hamiltonian is rotationally invariant, it cannot prefer one value of m over another. Hence the multiplet often appears as a degenerate set of energy states. An external magnetic field breaks the full rotational symmetry and can split the multiplet according to m, producing the Zeeman effect.

This multiplet language is one reason angular momentum theory is so economical. Rather than treating each magnetic state separately, one studies the whole representation labelled by l or j. The algebra then determines how the components of angular momentum connect the states inside the representation.

Interpreting l and m in different settings

The same angular momentum labels can carry different physical meanings depending on the system. For a particle in a central potential, l describes the angular part of the spatial wavefunction. It determines the eigenvalue of L^2, the number of angular nodes, and the centrifugal barrier in the radial equation. The quantum number m describes the projection along the chosen axis and distinguishes the $2l + 1$ magnetic substates.

For spin, the analogous labels are s and m_s. The value of s is fixed by the kind of particle or composite system being described. An electron always has $s = \frac{1}{2}$, while a spin-one particle always has $s = 1$. The magnetic quantum number m_s labels possible projections of that intrinsic angular momentum. There is no spatial spherical harmonic associated with s; the representation space is internal.

For total angular momentum, the labels are j and m_j. These may arise from adding orbital and spin angular momenta, or from adding angular momenta of several particles. The total quantum number j is often the best label when the Hamiltonian is rotationally invariant and includes couplings such as $\mathbf{L} \cdot \mathbf{S}$. Thus the symbols l, s, j are not interchangeable: they identify different angular momentum operators even though the algebraic spectrum has the same form.

Expectation values in $|l, m\rangle$

In a state $|l, m\rangle$, the expectation values of L_x and L_y vanish:

$$\langle L_x \rangle = \langle L_y \rangle = 0,$$

while

$$\langle L_z \rangle = m\hbar.$$

This does not mean L_x and L_y are zero. Their variances are generally nonzero. Since

$$L_x^2 + L_y^2 = L^2 - L_z^2,$$

we have

$$\langle L_x^2 + L_y^2 \rangle = \hbar^2[l(l+1) - m^2].$$

For states with rotational symmetry about the z-axis, the transverse uncertainty is shared equally:

$$\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{\hbar^2}{2}[l(l+1) - m^2].$$

This is a useful quantitative way to understand the vector model. The transverse components have zero mean but nonzero spread.

Large angular momentum and the classical limit

For large l, the spacing between neighboring m-values is still $\hbar$, but the total magnitude is of order $l\hbar$. Relative spacings become small, and the set of allowed projections begins to resemble a continuum. This is part of the classical limit. However, the exact quantum structure remains discrete for every finite l. The classical vector picture becomes accurate only when many closely spaced quantum states are involved or when one forms wave packets with large angular momentum.

This observation is useful in molecular rotation and semiclassical atomic physics. High rotational quantum numbers may be visualized almost classically, while low quantum numbers, especially $l = 0$, $l = 1$, and spin-$\frac{1}{2}$, require the full quantum interpretation.

Spherical harmonics as concrete eigenvectors

For orbital angular momentum, the abstract state $|l, m\rangle$ has a concrete wavefunction representative:

$$Y_l^m(\theta, \phi).$$

These functions are simultaneous eigenfunctions of L^2 and L_z. The m-dependence is simple:

$$Y_l^m(\theta, \phi) \propto P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi},$$

where P_l^m is an associated Legendre function. The factor $\mathrm{e}^{\mathrm{i}m\phi}$ gives the L_z eigenvalue, while the polar function makes the state an eigenfunction of L^2.

The ladder operators connect these functions:

$$L_\pm Y_l^m \propto Y_l^{m\pm1}.$$

This means the algebraic ladder is visible as a relation among angular wavefunctions. Starting from the highest state Y_l^l, one can lower repeatedly to generate all Y_l^m for the same l, up to normalization and phase. In this way the algebra and the differential-equation approach meet.

The lowest angular momentum state Y_0^0 is constant on the sphere. It has no angular nodes and corresponds to $l = 0$. The $l = 1$ harmonics form a three-dimensional space related to the Cartesian functions x/r, y/r, and z/r. The $l = 2$ harmonics form a five-dimensional space related to quadratic angular patterns. These connections explain why angular momentum labels are also shape labels in orbital diagrams.

Why m changes in integer steps

The unit step in m is built into the algebra through

$$[L_z, L_\pm] = \pm\hbar L_\pm.$$

If a state has L_z eigenvalue $m\hbar$, then $L_\pm$ produces a state with eigenvalue $(m \pm 1)\hbar$. There is no operator constructed from L_x and L_y that changes m by half a unit inside a given angular momentum representation. The multiplet is therefore evenly spaced in m.

This spacing has observable consequences. In a weak magnetic field, energy shifts are often proportional to m, so spectral lines split into evenly spaced components in the simplest Zeeman effect. More complicated situations modify the proportionality factor, but the underlying projection quantum number still comes from the angular momentum ladder.

1.5 Matrix Representation of Angular Momentum Operators

An operator becomes a matrix once a basis is chosen. For angular momentum the natural basis is the simultaneous eigenbasis of L^2 and L_z,

$$\{|l, m\rangle : m = l, l-1, \ldots, -l\}.$$

The ordering is a convention. In this chapter we order states from highest m to lowest m. In a fixed l subspace the dimension is $2l + 1$, so each angular momentum component is represented by a $(2l + 1) \times (2l + 1)$ matrix.

The matrix elements are defined by

$$(L_i)_{m'm} = \langle l, m'| L_i |l, m\rangle .$$

Because $L_z |l, m\rangle = m\hbar |l, m\rangle$, the L_z matrix is diagonal:

$$\langle l, m'| L_z |l, m\rangle = m\hbar\, \delta_{m'm}. \tag{1.53}$$

The ladder operators have nonzero matrix elements only between neighboring m values:

$$\langle l, m'| L_+ |l, m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\, \delta_{m',m+1}, \tag{1.54}$$

$$\langle l, m'| L_- |l, m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\, \delta_{m',m-1}. \tag{1.55}$$

Once L_+ and L_- are known, the Cartesian components follow from

$$L_x = \frac{1}{2}(L_+ + L_-), \qquad L_y = \frac{1}{2\mathrm{i}}(L_+ - L_-). \tag{1.56}$$

As a first example, consider a two-dimensional angular momentum representation with $j = \frac{1}{2}$. This representation is the one used for spin-$\frac{1}{2}$, so we write the basis as

$$|+\rangle \equiv |\tfrac{1}{2}, \tfrac{1}{2}\rangle , \qquad |-\rangle \equiv |\tfrac{1}{2}, -\tfrac{1}{2}\rangle .$$

In the ordered basis $(|+\rangle, |-\rangle)$,

$$J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.57}$$

The ladder operators are

$$J_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.58}$$

Therefore

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{1.59}$$

These matrices foreshadow the Pauli matrices, which will be studied in Sec. 1.7.

For $l = 1$, the basis is

$$(|1,1\rangle, |1,0\rangle, |1,-1\rangle).$$

Then

$$L_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.60}$$

The ladder operators are

$$L_+ = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \qquad L_- = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \tag{1.61}$$

Thus

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}. \tag{1.62}$$

One can check by direct multiplication that these matrices obey

$$[L_x, L_y] = \mathrm{i}\hbar L_z$$

and its cyclic partners. Also,

$$L_x^2 + L_y^2 + L_z^2 = 2\hbar^2 I_3,$$

which agrees with $l(l+1)\hbar^2 = 2\hbar^2$ for $l = 1$.

The matrix representation is not a different theory. It is simply the operator action written in a chosen basis. For example, the second column of the L_x matrix for $l = 1$ tells us how L_x acts on $|1,0\rangle$:

$$L_x|1,0\rangle = \frac{\hbar}{\sqrt{2}}|1,1\rangle + \frac{\hbar}{\sqrt{2}}|1,-1\rangle.$$

The column is the coordinate vector of the resulting state. The diagonal form of L_z says that L_z acts without mixing basis states, while L_x and L_y do mix them. This is the matrix expression of the fact that the chosen basis diagonalizes L_z, not L_x or L_y.

Normalization is also encoded in the matrices. The ladder coefficients were chosen so that if $|l,m\rangle$ is normalized, then the normalized state obtained after applying $L_\pm$ has unit norm after dividing by the coefficient. Hermiticity is visible as well: $L_x^\dagger = L_x$, $L_y^\dagger = L_y$, and $L_z^\dagger = L_z$, while $L_+^\dagger = L_-$.

Finite-dimensional matrices are especially useful when angular momentum is not represented by a spatial differential operator. Spin-$\frac{1}{2}$ has a two-dimensional Hilbert space, and its operators are 2×2 matrices. Coupled angular momenta can also be handled by matrices acting on product spaces. The abstract algebra, the ladder rules, and the matrix representation are therefore three views of the same structure.

A practical construction algorithm

For a fixed value of j, the matrices of J_x, J_y, J_z can be constructed by a simple algorithm. First choose the ordered basis

$$|j,j\rangle\,, |j,j-1\rangle\,, \ldots, |j,-j\rangle\,.$$

Second, write J_z as the diagonal matrix with entries $m\hbar$. Third, fill the nonzero entries of J_+ using

$$J_+\,|j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\,|j,m+1\rangle\,.$$

Fourth, set $J_- = J_+^\dagger$. Finally form J_x and J_y using

$$J_x = \frac{1}{2}(J_+ + J_-), \qquad J_y = \frac{1}{2\mathrm{i}}(J_+ - J_-).$$

This algorithm is often safer than trying to guess the matrices directly.

For example, for $j = \frac{3}{2}$ the dimension is four. The basis is

$$|\tfrac{3}{2},\tfrac{3}{2}\rangle\,, |\tfrac{3}{2},\tfrac{1}{2}\rangle\,, |\tfrac{3}{2},-\tfrac{1}{2}\rangle\,, |\tfrac{3}{2},-\tfrac{3}{2}\rangle\,.$$

The J_z matrix is diagonal with entries

$$\frac{3}{2}\hbar, \quad \frac{1}{2}\hbar, \quad -\frac{1}{2}\hbar, \quad -\frac{3}{2}\hbar.$$

The raising matrix has nonzero entries proportional to $\sqrt{3}, 2, \sqrt{3}$. These numbers are not mysterious; they are the square roots produced by the ladder formula. Larger angular momentum matrices are built in the same way.

Changing the measurement axis

The matrix representation also clarifies what happens when one measures a component different from the one used to define the basis. In the $|l,m\rangle$ basis, L_z is diagonal. A measurement of L_z therefore reads off the basis amplitudes directly. A measurement of L_x, however, requires diagonalizing the L_x matrix or expanding the state in the eigenbasis of L_x. The probabilities are not the squared moduli of the original m-basis coefficients unless the measured operator is L_z.

For spin-$\frac{1}{2}$, this is especially transparent. In the S_z basis,

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvectors of this matrix are equal superpositions of $|+\rangle$ and $|-\rangle$. Thus a spin prepared up along z has equal probabilities for $+\hbar/2$ and $-\hbar/2$ when measured along x. This is not a special feature of spin; it is the finite-dimensional version of incompatible angular momentum components.

Basis conventions and signs

When comparing matrices across textbooks, one must check basis ordering and phase conventions. If the basis is ordered from lowest m to highest m instead of highest to lowest, the matrices are rearranged by a permutation. If one changes the phase of a basis state,

$$|l,m\rangle \mapsto \mathrm{e}^{\mathrm{i}\alpha_m}\,|l,m\rangle\,,$$

the off-diagonal matrix elements of $L_\pm$ acquire compensating phase factors. Physical predictions do not change, but the displayed matrices may look different. The convention used here is

the standard Condon–Shortley convention, in which the ladder coefficients are chosen real and positive.

Matrices also make clear why L^2 is proportional to the identity in an irreducible fixed-l subspace:

$$L^2 = \hbar^2 l(l+1) I_{2l+1}.$$

This does not mean L^2 is trivial in the full Hilbert space. It means that once we restrict attention to one angular momentum multiplet, all states in that multiplet have the same total squared angular momentum.

Matrices as a laboratory for measurement

Finite matrices let us see measurement theory explicitly. Suppose a state in the $l = 1$ subspace is written as a column vector

$$|\psi\rangle = \begin{pmatrix} c_1 \\ c_0 \\ c_{-1} \end{pmatrix}$$

in the ordered basis $(|1,1\rangle, |1,0\rangle, |1,-1\rangle)$. A measurement of L_z gives $\hbar, 0, -\hbar$ with probabilities $|c_1|^2, |c_0|^2, |c_{-1}|^2$. A measurement of L_x requires the eigenvectors of the L_x matrix. The same column vector must be re-expanded in that eigenbasis before probabilities are read off.

This is a useful way to think about basis dependence. The state vector itself is an abstract object. Its column entries are coordinates in a chosen basis. Changing from the L_z eigenbasis to the L_x eigenbasis is like rotating the coordinate system in Hilbert space, but the probabilities for a particular measurement depend on the basis associated with that measurement.

The matrix picture also makes degeneracy concrete. Within a fixed l subspace, L^2 is proportional to the identity, so every vector in that subspace is an eigenvector of L^2 with the same eigenvalue. This means that an arbitrary superposition of $|l,m\rangle$ states with the same l still has definite L^2, even if it does not have definite L_z. For example,

$$\frac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$$

has definite $L^2 = 2\hbar^2$, but it is not an eigenstate of L_z. It may, however, be related to an oriented real p-orbital.

Why spin matrices appear so early

Although orbital angular momentum for a spatial wavefunction has only integer l, the $j = \frac{1}{2}$ matrix representation appears naturally because it is the smallest nontrivial representation of the angular momentum algebra. It cannot be realized as an ordinary single-valued scalar orbital wavefunction, but it is perfectly valid as an internal spin representation. This is why the same ladder construction produces the Pauli matrices. Matrix methods therefore reveal a possibility that coordinate wavefunctions alone might hide: angular momentum can be represented in internal spaces as well as in spatial motion.

1.6 Spin Angular Momentum

Spin is intrinsic angular momentum. It is intrinsic in the sense that it is not constructed from the position and momentum of a particle by $\mathbf{r} \times \mathbf{p}$. A particle can have spin even when it is

treated as pointlike, with no internal spatial extension in the model. Nevertheless, the spin components obey the angular momentum commutation relations

$$[S_i, S_j] = \mathrm{i}\hbar\, \epsilon_{ijk} S_k. \tag{1.63}$$

This is the essential reason spin is called angular momentum: it generates rotations in the internal spin space and contributes to total angular momentum.

Spin should not be visualized as a literal classical rotation of a tiny ball. Such a picture fails both quantitatively and conceptually. For the electron, a classical rotating charged sphere with the observed magnetic moment would require impossible surface speeds, and it would not naturally explain the two-valued structure observed in Stern–Gerlach experiments. The correct statement is more precise: spin is a quantum degree of freedom whose observables transform under rotations as angular momentum.

The spin eigenvalue equations are

$$S^2 \,|s, m_s\rangle = \hbar^2 s(s+1)\,|s, m_s\rangle\,, \tag{1.64}$$

$$S_z \,|s, m_s\rangle = m_s \hbar\,|s, m_s\rangle\,. \tag{1.65}$$

The spin quantum number s is fixed for a given elementary particle. An electron, proton, neutron, and many atoms have spin $s = \frac{1}{2}$; a photon has spin $s = 1$; some composite systems have integer spin and some have half-integer spin. For a fixed s, the allowed values of m_s are

$$m_s = -s, -s+1, \ldots, s-1, s.$$

The dimension of the spin Hilbert space is therefore $2s + 1$.

For spin-$\frac{1}{2}$, there are two basis states. Along the z-axis they are

$$|+\rangle \equiv |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle \equiv |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.66}$$

The notation $|+\rangle$ and $|-\rangle$ is shorthand for spin up and spin down along the chosen z-axis. The operator S_z acts as

$$S_z\,|+\rangle = \frac{\hbar}{2}\,|+\rangle\,, \qquad S_z\,|-\rangle = -\frac{\hbar}{2}\,|-\rangle\,.$$

Thus a measurement of S_z on $|+\rangle$ gives $+\hbar/2$ with certainty, while a measurement on $|-\rangle$ gives $-\hbar/2$ with certainty.

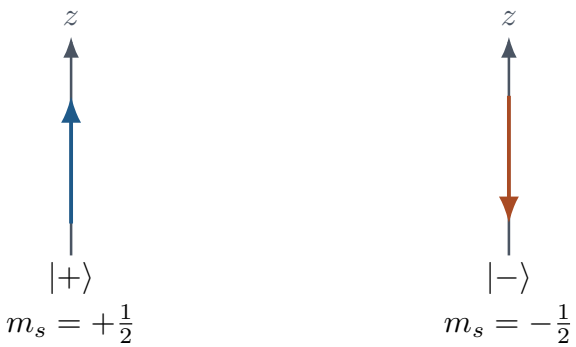

Figure 1.7: The two spin-$\frac{1}{2}$ basis states along the z-axis. The labels up and down refer to the sign of m_s, not to a classical rotating object.

A general spin-$\frac{1}{2}$ state is a two-component spinor,

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1. \tag{1.67}$$

If S_z is measured, the probability of obtaining $+\hbar/2$ is $|a|^2$, and the probability of obtaining $-\hbar/2$ is $|b|^2$. After the measurement, the spin state is projected onto the corresponding eigenstate, assuming an ideal projective measurement.

The experimental reality of spin is dramatically shown by the Stern–Gerlach experiment. A beam of neutral atoms with magnetic moments passes through an inhomogeneous magnetic field. Classically one might expect a continuous spread of deflections because a magnetic moment could have any projection along the field gradient. Instead, for spin-$\frac{1}{2}$ systems the beam splits into two discrete parts. The two spots correspond to the two possible values of m_s along the apparatus axis.

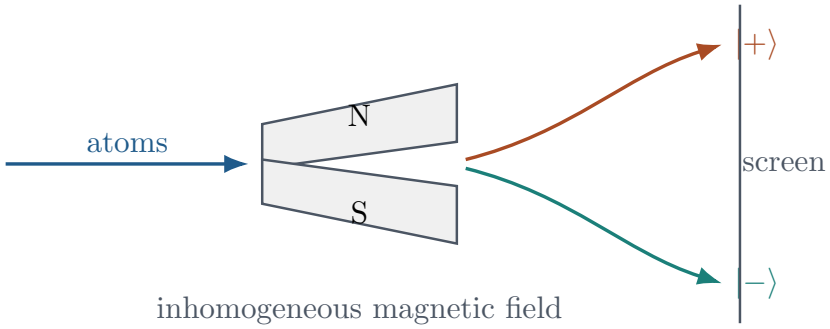

Figure 1.8: A Stern–Gerlach apparatus separates a spin-$\frac{1}{2}$ beam into two components. The apparatus axis defines the component of spin being measured.

The phrase "along the z-axis" deserves attention. The two-dimensional Hilbert space is the same no matter what axis is chosen, but the eigenvectors representing spin up and spin down depend on that axis. If the apparatus is rotated, it measures $\mathbf{S}\cdot\hat{\mathbf{n}}$ rather than S_z. A state that is definitely up along z is generally a superposition of up and down along x, and vice versa. This is the spin version of the noncommutativity already seen for orbital angular momentum.

Spin also contributes to magnetic moments. For an electron,

$$\boldsymbol{\mu}_S = -g_s \frac{e}{2m_e}\mathbf{S},$$

where $g_s \approx 2$. The minus sign appears because the electron charge is negative. This connection between spin and magnetic moment is what allows magnetic fields to measure or manipulate spin states. It is also central to magnetic resonance, spintronics, and atomic fine structure.

Spin therefore enlarges the angular momentum concept without abandoning the algebra. The same equations for eigenvalues, ladders, and rotations apply, but the representation space is internal. For spin-$\frac{1}{2}$, the relevant matrices are the Pauli matrices.

Sequential spin measurements

Spin-$\frac{1}{2}$ gives a clean laboratory for understanding incompatible observables. Suppose a beam is first passed through a Stern–Gerlach apparatus aligned along z, and only the spin-up output is kept. The state after this preparation is $|+\rangle$. If the beam is immediately sent through another

z-aligned apparatus, only the up output appears. This repeatability is what it means for $|+\rangle$ to be an eigenstate of S_z.

Now insert an x-aligned apparatus after the first z-preparation. The state $|+\rangle$ can be written as

$$|+\rangle = \frac{1}{\sqrt{2}}|+x\rangle + \frac{1}{\sqrt{2}}|-x\rangle .$$

The x-apparatus therefore splits the beam equally. If one keeps only the $+x$ output and then measures S_z again, the beam again splits equally. The intermediate S_x measurement has destroyed the definite S_z preparation. This is the operational meaning of $[S_x, S_z] \neq 0$.

Spin space is not coordinate space

A two-component spinor is sometimes written next to a spatial wavefunction, but the two factors represent different degrees of freedom. A complete electron state may have the form

$$\Psi(\mathbf{r}) = \begin{pmatrix} \psi_+(\mathbf{r}) \\ \psi_-(\mathbf{r}) \end{pmatrix} = \psi_+(\mathbf{r})\,|+\rangle + \psi_-(\mathbf{r})\,|-\rangle .$$

The variable $\mathbf{r}$ describes position in ordinary space. The two spin components describe amplitudes for spin up and spin down along the chosen axis. Orbital angular momentum differentiates the spatial dependence; spin matrices act on the two-component column. In a product notation, the total Hilbert space is

$$\mathcal{H}_{\text{space}} \otimes \mathcal{H}_{\text{spin}}.$$

This separation explains why L_i and S_j commute for a single electron:

$$[L_i, S_j] = 0.$$

They act on different factors of the Hilbert space. The total angular momentum is nevertheless their sum,

$$\mathbf{J} = \mathbf{L} + \mathbf{S},$$

because rotations affect both the spatial wavefunction and the spinor.

Magnetic moments and physical coupling

Spin would be far less visible experimentally if it did not couple to magnetic fields. For a spin-$\frac{1}{2}$ particle with magnetic moment $\boldsymbol{\mu}$, the interaction energy in a magnetic field is commonly written

$$H_B = -\boldsymbol{\mu} \cdot \mathbf{B}.$$

For the electron, $\boldsymbol{\mu}$ is proportional to $\mathbf{S}$, with a negative sign due to the electron's charge. In a uniform field along z, the spin energy is proportional to S_z, so the two spin states acquire different energies. In an inhomogeneous field, the force depends on the spin projection, which is what produces Stern–Gerlach splitting.

This magnetic coupling is one reason spin is not an optional mathematical decoration. It has directly measurable consequences: spectral line splitting, magnetic resonance frequencies, spin precession, and the magnetic structure of materials all depend on spin angular momentum.

What the Stern–Gerlach experiment really rules out

The Stern–Gerlach experiment is sometimes summarized by saying that spin is quantized. That is true, but the experiment teaches a more precise lesson. A classical magnetic moment entering an inhomogeneous field would experience a force depending on the continuous projection of the moment along the field gradient. A random beam of classical moments would therefore produce a continuous smear on the screen. A spin-$\frac{1}{2}$ beam produces two spots. The apparatus does not reveal a pre-existing continuum of orientations; it realizes a measurement with two eigenvalues.

The experiment also shows the role of state preparation. If the upper beam is selected and sent into a second identical apparatus, it does not split again. The first apparatus has prepared the state corresponding to spin up along its axis. But if the second apparatus is rotated, splitting reappears. This is exactly the behavior expected for noncommuting spin components. The sequence of measurements is often more revealing than a single measurement because it shows both quantization and incompatibility.

Spin also demonstrates that quantum states are not merely statements of ignorance about classical variables. A spin-up state along z is not a statistical mixture of particles secretly up or down along x. It is a coherent superposition in the x-basis. This distinction can be tested by recombining beams and observing interference, though the idealized analysis requires careful control of phases. The two-component spinor carries phase information that a classical probability distribution would not contain.

Spin and identical particles

Spin has another deep role in quantum mechanics: it is tied to exchange symmetry. Particles with half-integer spin are fermions, and identical fermions have antisymmetric total wavefunctions. Particles with integer spin are bosons, and identical bosons have symmetric total wavefunctions. This spin-statistics connection is a relativistic result, but its consequences appear throughout nonrelativistic quantum mechanics. The electronic structure of atoms, the Pauli exclusion principle, and the stability of matter all depend on electron spin-$\frac{1}{2}$.

When two electrons are considered together, the spin state may be a symmetric triplet or an antisymmetric singlet. The spatial part of the two-electron wavefunction must then have the opposite exchange symmetry for the total fermionic state to be antisymmetric. Thus spin affects not only magnetic properties but also spatial structure and chemical bonding.

Spin precession

A spin in a magnetic field generally precesses. If the Hamiltonian is proportional to S_z,

$$H = \omega S_z,$$

then the states $|+\rangle$ and $|-\rangle$ acquire different phases in time:

$$|\psi(t)\rangle = a\mathrm{e}^{-\mathrm{i}\omega t/2}\,|+\rangle + b\mathrm{e}^{\mathrm{i}\omega t/2}\,|-\rangle\,.$$

The relative phase changes at angular frequency ω. On the Bloch sphere this appears as precession about the z-axis. This simple time evolution is the basis of magnetic resonance and many qubit-control protocols.

1.7 Pauli Spin Matrices and Spin Eigenvectors

For spin-$\frac{1}{2}$, the spin operators are represented by the Pauli matrices:

$$S_i = \frac{\hbar}{2}\sigma_i, \qquad i = x, y, z, \tag{1.68}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.69}$$

These three matrices, together with the identity, form a basis for all 2×2 complex matrices. They are Hermitian and traceless, and each squares to the identity:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I.$$

The spin operators are therefore Hermitian, as required for observables.

The Pauli matrices satisfy two important algebraic relations. Their commutator is

$$[\sigma_i, \sigma_j] = 2\mathrm{i}\,\epsilon_{ijk}\sigma_k, \tag{1.70}$$

which implies

$$[S_i, S_j] = \frac{\hbar^2}{4}\,[\sigma_i, \sigma_j] = \mathrm{i}\hbar\,\epsilon_{ijk}S_k.$$

Thus the Pauli matrices give the spin-$\frac{1}{2}$ representation of the angular momentum algebra. Their anticommutator is

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}I. \tag{1.71}$$

This second relation is special to spin-$\frac{1}{2}$ and is extremely useful in calculations involving magnetic fields, relativistic spinors, and two-level quantum systems.

The eigenvectors of σ_z are already familiar:

$$\sigma_z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \sigma_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.72}$$

Thus S_z has eigenvalues $+\hbar/2$ and $-\hbar/2$.

For σ_x, solve

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.$$

The equations are $b = \lambda a$ and $a = \lambda b$, so $\lambda^2 = 1$. The normalized eigenvectors are

$$|{+x}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |{-x}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1.73}$$

Therefore

$$S_x\,|{+x}\rangle = \frac{\hbar}{2}\,|{+x}\rangle, \qquad S_x\,|{-x}\rangle = -\frac{\hbar}{2}\,|{-x}\rangle.$$

For σ_y, the eigenvalue equation gives

$$-\mathrm{i}b = \lambda a, \qquad \mathrm{i}a = \lambda b.$$

Again $\lambda = \pm 1$. A convenient normalized choice is

$$|{+y}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \qquad |{-y}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}. \tag{1.74}$$

The relative phase between the two components is essential. It distinguishes a spin state pointing along y from one pointing along x.

A general normalized spinor can be written

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle\,, \qquad |a|^2 + |b|^2 = 1. \tag{1.75}$$

If S_z is measured, the probabilities are

$$P(+\hbar/2) = |a|^2, \qquad P(-\hbar/2) = |b|^2.$$

If S_x is measured, we must expand $|\psi\rangle$ in the $\{|+x\rangle\,, |-x\rangle\}$ basis. The probability of $+\hbar/2$ along x is

$$P(+x) = |\,\langle +x|\,\psi\rangle|^2 = \frac{1}{2}|a+b|^2. \tag{1.76}$$

This illustrates a general principle: probabilities depend on the overlap between the prepared state and the eigenstates of the measured observable.

For an arbitrary unit vector

$$\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

the spin component along $\hat{\mathbf{n}}$ is

$$\mathbf{S}\cdot\hat{\mathbf{n}} = \frac{\hbar}{2}\,\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}, \tag{1.77}$$

where

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} = \begin{pmatrix} \cos\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta & -\cos\theta \end{pmatrix}. \tag{1.78}$$

The eigenvalues of $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$ are always $+1$ and -1, so the eigenvalues of $\mathbf{S}\cdot\hat{\mathbf{n}}$ are always $+\hbar/2$ and $-\hbar/2$. A normalized eigenvector with eigenvalue $+1$ is

$$|+\hat{\mathbf{n}}\rangle = \begin{pmatrix} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2) \end{pmatrix}. \tag{1.79}$$

The half-angles are not a typographical accident. They are a signature of the spinor representation of rotations. A spinor changes sign under a 2π rotation and returns exactly to itself only after a 4π rotation. Physical probabilities are unchanged by the overall sign, but the sign matters in interference.

The Bloch sphere is a useful visualization of Eq. (1.79). Up to an irrelevant overall phase, every normalized spin-$\frac{1}{2}$ state corresponds to a point on a unit sphere. The north pole represents $|+\rangle$, the south pole represents $|-\rangle$, and points on the equator represent equal-amplitude superpositions with different relative phases. The Bloch sphere is not ordinary physical space; it is a geometric representation of the two-dimensional spin Hilbert space.

Pauli matrix identities

For spin-$\frac{1}{2}$,

$$S_i = \frac{\hbar}{2}\sigma_i, \qquad [\sigma_i, \sigma_j] = 2\mathrm{i}\epsilon_{ijk}\sigma_k, \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}I.$$

Measurements along any direction have outcomes $\pm\hbar/2$.

Products of Pauli matrices

The commutator and anticommutator identities can be combined into the compact product formula

$$\sigma_i \sigma_j = \delta_{ij} I + \mathrm{i}\epsilon_{ijk}\sigma_k. \tag{1.80}$$

This relation is one of the most efficient tools in spin-$\frac{1}{2}$ calculations. For two ordinary vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) &= a_i b_j \sigma_i \sigma_j \\ &= a_i b_i I + \mathrm{i}\epsilon_{ijk} a_i b_j \sigma_k \\ &= (\mathbf{a}\cdot\mathbf{b})I + \mathrm{i}\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b}).\end{aligned} \tag{1.81}$$

This identity is used repeatedly in two-level systems, spin rotations, and relativistic quantum mechanics.

As a simple consequence, if $\hat{\mathbf{n}}$ is a unit vector,

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})^2 = I.$$

Therefore the eigenvalues of $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$ must be ± 1. This gives an algebraic explanation of why spin-$\frac{1}{2}$ measurements along any axis have the same two possible results.

Expectation values and the Bloch vector

For a normalized spinor $|\psi\rangle$, define the three expectation values

$$\langle\sigma_x\rangle, \qquad \langle\sigma_y\rangle, \qquad \langle\sigma_z\rangle.$$

These form a real vector

$$\mathbf{r}_\psi = \langle\boldsymbol{\sigma}\rangle.$$

For a pure spin-$\frac{1}{2}$ state this vector has unit length. It is the Bloch vector associated with the state. If

$$|\psi\rangle = \begin{pmatrix} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2) \end{pmatrix},$$

then

$$\langle\boldsymbol{\sigma}\rangle = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Thus the angles θ, ϕ in the spinor correspond to the point on the Bloch sphere.

The expectation value of the spin vector is

$$\langle\mathbf{S}\rangle = \frac{\hbar}{2}\langle\boldsymbol{\sigma}\rangle.$$

This resembles a classical vector, but it should not be confused with a state of definite spin components. A state with Bloch vector along z has definite S_z, but it has uncertain S_x and S_y. The Bloch vector summarizes expectation values, not simultaneous hidden values of all components.

Spin rotations with Pauli matrices

For spin-$\frac{1}{2}$, a rotation by angle θ about $\hat{\mathbf{n}}$ is

$$U(\hat{\mathbf{n}}, \theta) = \exp\left(-\frac{\mathrm{i}\theta}{2}\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\right). \tag{1.82}$$

Using $(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})^2 = I$, the exponential can be evaluated exactly:

$$U(\hat{\mathbf{n}}, \theta) = \cos\frac{\theta}{2}\, I - \mathrm{i}\sin\frac{\theta}{2}\,\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}. \tag{1.83}$$

The half-angle is the mathematical reason spinors acquire a minus sign under a 2π rotation:

$$U(\hat{\mathbf{n}}, 2\pi) = -I.$$

After a 4π rotation the spinor returns to itself. This behavior is not visible in ordinary classical vectors but is experimentally meaningful in interference phenomena.

Density matrix viewpoint

Although this chapter mainly uses state vectors, spin-$\frac{1}{2}$ systems are also often described by density matrices. Any spin-$\frac{1}{2}$ density matrix can be written as

$$\rho = \frac{1}{2}(I + \mathbf{r}\cdot\boldsymbol{\sigma}),$$

where $\mathbf{r}$ is a real vector with $|\mathbf{r}| \leq 1$. Pure states have $|\mathbf{r}| = 1$, while mixed states have $|\mathbf{r}| < 1$. This form is useful because measurement probabilities along $\hat{\mathbf{n}}$ are

$$P(+\hat{\mathbf{n}}) = \frac{1}{2}(1 + \mathbf{r}\cdot\hat{\mathbf{n}}), \qquad P(-\hat{\mathbf{n}}) = \frac{1}{2}(1 - \mathbf{r}\cdot\hat{\mathbf{n}}).$$

The formula shows directly how the geometry of the Bloch sphere encodes spin measurement statistics.

Relative phase is observable

In a spinor

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle\,,$$

the overall phase is not observable: $|\psi\rangle$ and $\mathrm{e}^{\mathrm{i}\alpha}\,|\psi\rangle$ represent the same physical state. The relative phase between a and b, however, is observable. It affects measurements along axes other than z. For example,

$$\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

is spin up along x, while

$$\frac{1}{\sqrt{2}}(|+\rangle + \mathrm{i}\,|-\rangle)$$

is spin up along y. Both states give equal probabilities for $S_z = \pm\hbar/2$, but they are distinguished by S_x and S_y measurements.

This is one of the most important lessons of spin-$\frac{1}{2}$. Probabilities in one basis do not fully specify the state. Relative phases matter because quantum amplitudes, not probabilities alone, are the objects that transform between bases. The Pauli matrices make this concrete: σ_x is sensitive to the real part of a^*b, while σ_y is sensitive to the imaginary part.

Pauli matrices as observables and generators

Each Pauli matrix plays two related roles. As an observable, σ_i represents twice the spin component $S_i/\hbar$. Its eigenvectors are the states with definite spin along the corresponding axis. As a generator, $\sigma_i/2$ appears in the rotation operator. This dual role mirrors the broader role of angular momentum: the same operator is both measurable and responsible for transformations.

For example, σ_z measures spin along z, but it also generates rotations about z:

$$U_z(\phi) = \exp(-\mathrm{i}\phi\sigma_z/2).$$

Acting on a spinor, this gives

$$U_z(\phi)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2}a \\ \mathrm{e}^{\mathrm{i}\phi/2}b \end{pmatrix}.$$

The relative phase changes by ϕ, which corresponds to a rotation of the Bloch vector about the z-axis.

Two-level systems beyond literal spin

The Pauli matrix formalism applies to many two-level quantum systems that are not literally spin. A two-level atom, a qubit in a superconducting circuit, a photon polarization subspace, or a pair of localized states in a double well can all be described using Pauli matrices after choosing a basis. In such cases the word "spin" may be used metaphorically as pseudospin. The mathematics is the same because any two-dimensional Hilbert space supports the Pauli matrix basis.

This broad usefulness is one reason the spin-$\frac{1}{2}$ formalism is central to modern quantum physics. It is the simplest nontrivial Hilbert space, yet it contains superposition, incompatible measurements, phase, rotations, and unitary dynamics. Angular momentum theory supplies the physical interpretation when the two-level system is a real spin, and it supplies a powerful analogy when the two levels represent something else.

Thought experiment: changing measurement axes

Consider a spin prepared in the state $|+\rangle$, spin up along z. A measurement of S_z is certain to give $+\hbar/2$. A measurement of S_x, however, is described by the basis

$$|+x\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \qquad |-x\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

Rewriting,

$$|+\rangle = \frac{1}{\sqrt{2}}(|+x\rangle + |-x\rangle),$$

so the two S_x outcomes are equally probable. If the $+x$ outcome is selected, the state becomes $|+x\rangle$. A subsequent S_z measurement again has equal probabilities for up and down.

This sequence is a compact illustration of quantum state update. Measurement is not merely the passive revelation of a pre-existing component. It prepares an eigenstate of the measured observable. Since S_x and S_z are incompatible, preparation by one measurement changes the statistics of the other.

Now compare the state

$$|+y\rangle = \frac{1}{\sqrt{2}}(|+\rangle + \mathrm{i}\,|-\rangle).$$

It also gives equal probabilities for S_z, but it is not the same as $|+x\rangle$. The difference is the relative phase. Measuring S_y distinguishes them perfectly: $|+y\rangle$ is an eigenstate of S_y, while $|+x\rangle$ is not. This shows why a quantum state contains more information than a list of probabilities for one measurement basis.

Hamiltonians written with Pauli matrices

Any two-level Hamiltonian can be written as

$$H = a_0 I + \mathbf{a} \cdot \boldsymbol{\sigma},$$

where a_0 is a real energy offset and $\mathbf{a}$ is a real vector. The offset shifts both energies equally. The vector $\mathbf{a}$ determines the energy splitting and the preferred measurement axis. The eigenstates are spinors aligned and antialigned with $\mathbf{a}$ on the Bloch sphere.

For a spin in a magnetic field this form is not just mathematical:

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}$$

is proportional to $\boldsymbol{\sigma} \cdot \mathbf{B}$. The spin precesses about the direction of $\mathbf{B}$, and the energy eigenstates are spin up and down along the field. This is the basic structure behind magnetic resonance and qubit control. The Pauli matrices therefore provide both a measurement language and a dynamics language for two-state systems.

1.8 Addition of Angular Momenta

Many quantum systems contain more than one angular momentum. An electron in an atom has orbital angular momentum $\mathbf{L}$ and spin angular momentum $\mathbf{S}$. Two spin-$\frac{1}{2}$ particles have two spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$. A multi-electron atom may have several orbital and spin angular momenta that combine in different coupling schemes. The common problem is to construct the total angular momentum

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2. \tag{1.84}$$

The components are $J_i = J_{1i} + J_{2i}$. If the two angular momenta act on different Hilbert spaces, then every component of $\mathbf{J}_1$ commutes with every component of $\mathbf{J}_2$:

$$[J_{1i}, J_{2j}] = 0.$$

It follows that $\mathbf{J}$ itself obeys the angular momentum algebra:

$$[J_i, J_j] = \mathrm{i}\hbar\, \epsilon_{ijk} J_k.$$

There are two natural bases for the combined system. The uncoupled basis diagonalizes the individual squared angular momenta and individual z-components:

$$|j_1, m_1\rangle\, |j_2, m_2\rangle\,. \tag{1.85}$$

This basis is convenient when the Hamiltonian depends separately on J_{1z} and J_{2z}, or when the two subsystems are measured independently. The coupled basis diagonalizes the total angular momentum:

$$|j, m\rangle \quad \text{with fixed } j_1, j_2 \text{ understood.} \tag{1.86}$$

More fully one may write $|j_1, j_2; j, m\rangle$. This basis is convenient when the Hamiltonian contains $\mathbf{J}^2$ or $\mathbf{J}_1 \cdot \mathbf{J}_2$.

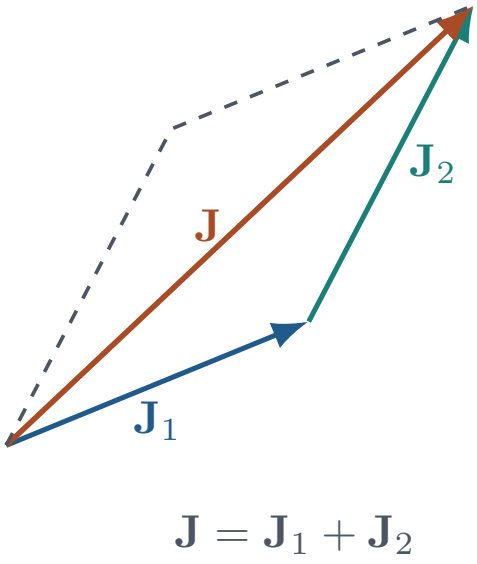

Figure 1.9: Classically angular momenta add as vectors. Quantum mechanically the allowed total angular momentum quantum numbers are restricted by the angular momentum algebra.

The allowed total angular momentum quantum numbers are

$$j = |j_1 - j_2|,\ |j_1 - j_2| + 1, \ldots,\ j_1 + j_2. \tag{1.87}$$

For each allowed j, the magnetic quantum number is

$$m = -j, -j+1, \ldots, j.$$

In the uncoupled basis the total z-component is

$$J_z = J_{1z} + J_{2z},$$

so

$$m = m_1 + m_2. \tag{1.88}$$

This selection rule is simple but powerful: only uncoupled states with the correct sum of magnetic quantum numbers can contribute to a given coupled state.

A useful check is the equality of dimensions:

$$(2j_1 + 1)(2j_2 + 1) = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j + 1). \tag{1.89}$$

The left-hand side is the number of uncoupled product states. The right-hand side is the number of coupled states. No states are lost and no extra states are created; the coupled basis is a different basis for the same Hilbert space.

The most important example is the addition of two spin-$\frac{1}{2}$ angular momenta. Here $j_1 = j_2 = \frac{1}{2}$, so the allowed total values are

$$j = 1, \quad j = 0.$$

The product space has dimension $2 \times 2 = 4$. The coupled states consist of a triplet with $j = 1$, containing three states, and a singlet with $j = 0$, containing one state.

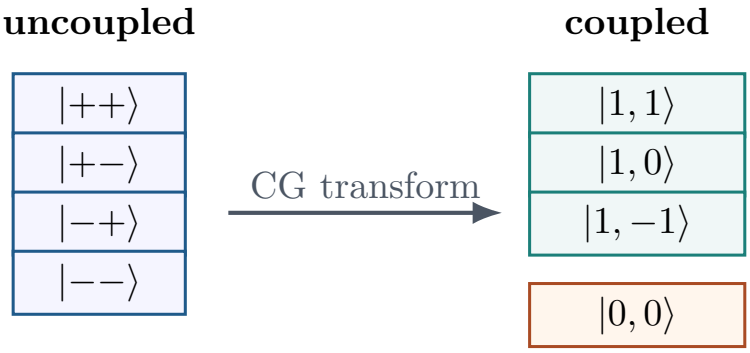

Figure 1.10: The uncoupled basis labels individual angular momentum projections. The coupled basis labels total angular momentum. Clebsch–Gordan coefficients perform the transformation between them.

Start with the highest possible m. The only product state with $m = m_1 + m_2 = 1$ is

$$|1,1\rangle = |+\rangle\,|+\rangle\,. \tag{1.90}$$

Apply the total lowering operator

$$J_- = S_{1-} + S_{2-}.$$

On one hand,

$$J_-\,|1,1\rangle = \hbar\sqrt{1(1+1) - 1(1-1)}\;|1,0\rangle = \hbar\sqrt{2}\;|1,0\rangle\,.$$

On the other hand,

$$(S_{1-} + S_{2-})\,|+\rangle\,|+\rangle = \hbar\,|-\rangle\,|+\rangle + \hbar\,|+\rangle\,|-\rangle\,.$$

Therefore

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle\,|-\rangle + |-\rangle\,|+\rangle\right). \tag{1.91}$$

Lowering once more gives

$$|1,-1\rangle = |-\rangle\,|-\rangle\,. \tag{1.92}$$

These three states are symmetric under interchange of the two spins.

The remaining $m = 0$ product-state combination must be orthogonal to $|1,0\rangle$. Up to an overall phase, it is

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle\,|-\rangle - |-\rangle\,|+\rangle\right). \tag{1.93}$$

This is the singlet state, antisymmetric under interchange. It has total angular momentum zero:

$$J^2\,|0,0\rangle = 0.$$

Although each particle individually has spin $\frac{1}{2}$, the pair as a whole has no net angular momentum in the singlet state.

Two spin-$\frac{1}{2}$ states

$$
\begin{aligned}
|1,1\rangle &= |+\rangle\,|+\rangle\,, \\
|1,0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle\,|-\rangle + |-\rangle\,|+\rangle), \\
|1,-1\rangle &= |-\rangle\,|-\rangle\,, \\
|0,0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle\,|-\rangle - |-\rangle\,|+\rangle).
\end{aligned}
$$

The first three form the symmetric triplet. The last is the antisymmetric singlet.

The symmetric and antisymmetric character becomes crucial when the particles are identical. For identical fermions, the total state must be antisymmetric under exchange. For identical bosons, it must be symmetric. Spin coupling therefore affects the allowed spatial states. This is one of the routes by which angular momentum enters chemistry, condensed matter physics, and quantum information.

The addition rules also explain the pattern of total atomic angular momentum. If an electron has orbital angular momentum l and spin $s = \frac{1}{2}$, then

$$j = l + \frac{1}{2} \quad \text{or} \quad j = l - \frac{1}{2}$$

for $l > 0$. For $l = 0$, only $j = \frac{1}{2}$ is possible. These total angular momentum values organize fine-structure levels and selection rules in atoms.

Compatible operators in the two bases

The distinction between coupled and uncoupled bases can be summarized by listing complete commuting sets. In the uncoupled basis, the natural commuting operators are

$$J_1^2, \quad J_2^2, \quad J_{1z}, \quad J_{2z}.$$

Their simultaneous eigenstates are $|j_1, m_1\rangle\,|j_2, m_2\rangle$. In the coupled basis, the natural commuting operators are

$$J_1^2, \quad J_2^2, \quad J^2, \quad J_z.$$

The values of j_1 and j_2 are retained, but the individual projections m_1, m_2 are replaced by total j, m. The two bases are complete descriptions of the same space, but they diagonalize different observables.

This matters because a Hamiltonian chooses which basis is convenient. A Hamiltonian with a term proportional to $J_{1z} + J_{2z}$ is diagonal in either basis with respect to m, but a term proportional to $\mathbf{J}_1 \cdot \mathbf{J}_2$ is simplest in the coupled basis:

$$\mathbf{J}_1 \cdot \mathbf{J}_2 = \frac{1}{2}(J^2 - J_1^2 - J_2^2).$$

By contrast, if two spins are placed in different local magnetic fields, the uncoupled basis may be more natural.

The triangle rule

The allowed range

$$|j_1 - j_2| \le j \le j_1 + j_2$$

is often called the triangle rule. Its classical analogue is the statement that the magnitude of the sum of two vectors lies between the difference and the sum of their magnitudes. Quantum mechanically, the rule is discrete: j changes in integer steps. Thus adding $j_1 = 2$ and $j_2 = 1$ gives $j = 1, 2, 3$, while adding $j_1 = 2$ and $j_2 = \frac{1}{2}$ gives $j = \frac{3}{2}, \frac{5}{2}$.

The dimension check gives a quick consistency test. For $j_1 = 2$ and $j_2 = 1$, the uncoupled space has dimension $5 \times 3 = 15$. The coupled multiplets have dimensions

$$(2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 3 + 1) = 3 + 5 + 7 = 15.$$

Every product state is accounted for.

Example: orbital plus spin

Consider one electron with orbital angular momentum $l = 1$ and spin $s = \frac{1}{2}$. The uncoupled space has dimension

$$(2l + 1)(2s + 1) = 3 \times 2 = 6.$$

The allowed total angular momenta are

$$j = \frac{3}{2}, \quad \frac{1}{2}.$$

The corresponding dimensions are $2j + 1 = 4$ and 2, which again sum to 6. In atomic notation, these are the $p_{3/2}$ and $p_{1/2}$ levels. Spin-orbit coupling separates them because $\mathbf{L} \cdot \mathbf{S}$ has different expectation values for the two j values.

This example is the prototype for much of atomic spectroscopy. The uncoupled notation $|l, m_l\rangle\,|s, m_s\rangle$ is intuitive when thinking about orbital and spin projections separately. The coupled notation $|l, s; j, m_j\rangle$ is the one adapted to total angular momentum and spin-orbit interactions.

How to choose a basis in applications

A useful rule is to choose the basis that diagonalizes the largest or most important part of the Hamiltonian. If two angular momenta interact strongly through a scalar product, the coupled basis is usually best because $\mathbf{J}_1 \cdot \mathbf{J}_2$ is diagonal in j. If the angular momenta are affected separately by external fields, the uncoupled basis may be better. In perturbation theory one often changes basis as different terms become important.

For example, in weak spin-orbit coupling one may start from uncoupled orbital and spin states and then form coupled states to diagonalize the perturbation. In a strong external magnetic field, the field may decouple $\mathbf{L}$ and $\mathbf{S}$ enough that m_l and m_s become more useful labels than j, m_j. This is the physical content behind different coupling schemes in spectroscopy.

Addition is not ordinary vector addition of sharp arrows

The notation $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ resembles classical vector addition, but quantum addition has extra structure. In a coupled state $|j, m\rangle$, the total J^2 and J_z are sharp, but the individual projections m_1, m_2 need not be sharp. For example, the triplet state

$$|1, 0\rangle = \frac{1}{\sqrt{2}}(|+\rangle\,|-\rangle + |-\rangle\,|+\rangle)$$

has definite total $j = 1, m = 0$, yet neither particle separately has a definite S_z value before measurement. The total projection is sharp because both terms have $m_1 + m_2 = 0$, but the individual projections are entangled.

The singlet state is even more striking. It has total angular momentum zero, but each particle still has spin-$\frac{1}{2}$. Measurements of the two spins along the same axis are perfectly anticorrelated. The state is rotationally invariant in the sense that it remains a singlet no matter what common axis is chosen. This rotational invariance of the singlet is a cornerstone of many discussions of quantum entanglement.

Counting states as a consistency test

Whenever angular momenta are added, counting dimensions is a powerful way to catch mistakes. The product basis dimension is $(2j_1 + 1)(2j_2 + 1)$. The coupled multiplets must account for exactly this number of states. If a proposed list of allowed j values gives the wrong total dimension, the list is incomplete or contains an impossible value. This check is simple, but it prevents many errors in angular momentum coupling problems.

1.9 Clebsch–Gordan Coefficients

Clebsch–Gordan coefficients are the numbers that connect the uncoupled and coupled angular momentum bases. They are defined by the expansion

$$|j_1, j_2; j, m\rangle = \sum_{m_1, m_2} C^{j\,m}_{j_1 m_1\, j_2 m_2}\, |j_1, m_1\rangle\, |j_2, m_2\rangle\,. \tag{1.94}$$

The coefficient

$$C^{j\,m}_{j_1 m_1\, j_2 m_2} = \langle j_1, m_1; j_2, m_2 \rangle\, j_1, j_2; j, m$$

is the amplitude that the coupled state $|j_1, j_2; j, m\rangle$ is found in the uncoupled product state $|j_1, m_1\rangle\, |j_2, m_2\rangle$.

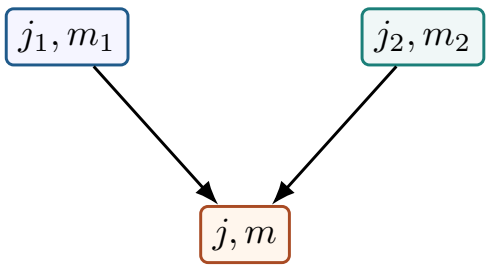

Figure 1.11: A Clebsch–Gordan coefficient specifies how two angular momenta j_1 and j_2 combine to a state of good total angular momentum j.

The first selection rule is

$$m = m_1 + m_2. \tag{1.95}$$

If this condition is not satisfied, the coefficient is zero. This follows because $J_z = J_{1z} + J_{2z}$, so an uncoupled product state has total magnetic quantum number $m_1 + m_2$. A coupled state with magnetic quantum number m cannot contain components with a different J_z eigenvalue.

The coefficients also obey normalization:

$$\sum_{m_1,m_2} \left| C^{j\,m}_{j_1 m_1\, j_2 m_2} \right|^2 = 1, \tag{1.96}$$

for each fixed coupled state. Orthogonality of different coupled states gives

$$\sum_{m_1,m_2} C^{j\,m}_{j_1 m_1\, j_2 m_2} C^{j' m'}_{j_1 m_1\, j_2 m_2} = \delta_{jj'}\delta_{mm'}, \tag{1.97}$$

assuming real coefficients in the standard phase convention. More generally one includes complex conjugation on one coefficient, but standard Clebsch–Gordan tables are real.

For two spin-$\frac{1}{2}$ particles, the coefficients can be read from the triplet and singlet states:

coupled state	uncoupled expansion
$\lvert 1,1\rangle$	$\lvert+\rangle\lvert+\rangle$
$\lvert 1,0\rangle$	$\frac{1}{\sqrt{2}}\lvert+\rangle\lvert-\rangle + \frac{1}{\sqrt{2}}\lvert-\rangle\lvert+\rangle$
$\lvert 1,-1\rangle$	$\lvert-\rangle\lvert-\rangle$
$\lvert 0,0\rangle$	$\frac{1}{\sqrt{2}}\lvert+\rangle\lvert-\rangle - \frac{1}{\sqrt{2}}\lvert-\rangle\lvert+\rangle$

For example,

$$C^{1\,0}_{\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,-\frac{1}{2}} = \frac{1}{\sqrt{2}}, \qquad C^{0\,0}_{\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,-\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

The same uncoupled state $\lvert+\rangle\lvert-\rangle$ contributes to both the triplet $m=0$ state and the singlet $m=0$ state, but with different relative signs when paired with $\lvert-\rangle\lvert+\rangle$.

Why are these coefficients useful? Many Hamiltonians are simple in the coupled basis. Suppose a two-spin Hamiltonian contains the scalar product

$$\mathbf{S}_1 \cdot \mathbf{S}_2.$$

Using

$$\mathbf{J}^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2 = S_1^2 + S_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2,$$

we have

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}(J^2 - S_1^2 - S_2^2).$$

Thus the coupled basis diagonalizes the interaction. In the triplet state $j=1$, the eigenvalue is $\hbar^2/4$. In the singlet state $j=0$, it is $-3\hbar^2/4$. The Clebsch–Gordan coefficients then tell us how those eigenstates appear in terms of individual spin-up and spin-down states.

In atomic physics, a central example is l-s coupling for an electron:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}. \tag{1.98}$$

The uncoupled basis is

$$\lvert l, m_l\rangle \, \lvert \tfrac{1}{2}, m_s\rangle .$$

The coupled basis is

$$\lvert l, \tfrac{1}{2}; j, m_j\rangle ,$$

with $j = l \pm \frac{1}{2}$ and $m_j = m_l + m_s$. Spin-orbit interactions are naturally expressed in terms of $\mathbf{L}\cdot\mathbf{S}$, and therefore in terms of J^2, L^2, and S^2:

$$\mathbf{L} \cdot \mathbf{S} = \frac{1}{2}(J^2 - L^2 - S^2).$$

The coupled basis is the natural basis for such interactions.

It is worth emphasizing that Clebsch–Gordan coefficients are not new physical postulates. They are basis transformation coefficients forced by angular momentum algebra. Once a phase convention is fixed, the coefficients are determined by normalization, orthogonality, the highest-weight state, and repeated application of lowering operators. Tables are convenient, but the logic is simple: construct states of definite total J^2 and J_z out of product states of definite J_{1z} and J_{2z}.

The coefficients also encode selection rules. If a matrix element involves a rotationally invariant or tensor operator, many terms vanish because the angular momenta cannot combine in the required way. In spectroscopy, this algebra greatly reduces the number of possible transitions. In scattering and nuclear physics, it allows states to be classified by total angular momentum even when the underlying motion is complicated.

Meaning of a Clebsch–Gordan coefficient

$$C^{j\,m}_{j_1 m_1\, j_2 m_2} = \langle j_1, m_1; j_2, m_2 \rangle\, j_1, j_2; j, m$$

is a probability amplitude for finding the coupled state $|j_1, j_2; j, m\rangle$ in the uncoupled product state $|j_1, m_1\rangle\,|j_2, m_2\rangle$. It is nonzero only if $m = m_1 + m_2$.

How coefficients are generated

Clebsch–Gordan coefficients can be calculated systematically. The most transparent method begins with the highest-weight state. For given j_1 and j_2, the state with maximum $m = j_1 + j_2$ is unique:

$$|j_1 + j_2, j_1 + j_2\rangle = |j_1, j_1\rangle\,|j_2, j_2\rangle\,.$$

This fixes the top coefficient to be one in the standard phase convention. Applying the total lowering operator

$$J_- = J_{1-} + J_{2-}$$

then generates the rest of the $j = j_1 + j_2$ multiplet. After that, the highest remaining state with the next value of j is constructed by orthogonality to the states already found. Repeating this process constructs the full table.

This procedure is conceptually more important than memorizing tables. It shows that the coefficients are forced by the angular momentum algebra. Orthogonality determines relative signs; normalization determines magnitudes; the ladder operators determine recursion relations. Tables are useful because the arithmetic becomes tedious, not because the coefficients are independent data.

Orthogonality in both directions

The coupled and uncoupled bases are both orthonormal and complete. Therefore the matrix of Clebsch–Gordan coefficients is unitary. This gives two complementary orthogonality relations. One was already written in Eq. (1.97). The other says that uncoupled states are also recovered completely:

$$\sum_{j,m} C^{j\,m}_{j_1 m_1\, j_2 m_2} C^{j\,m}_{j_1 m_1'\, j_2 m_2'} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}. \tag{1.99}$$

This equation is simply the statement that the change of basis can be inverted. In practice, it lets us write product states as sums of coupled states. For two spin-$\frac{1}{2}$ particles,

$$|+\rangle\,|-\rangle = \frac{1}{\sqrt{2}}\,|1,0\rangle + \frac{1}{\sqrt{2}}\,|0,0\rangle\,,$$

while

$$|-\rangle\,|+\rangle = \frac{1}{\sqrt{2}}\,|1,0\rangle - \frac{1}{\sqrt{2}}\,|0,0\rangle\,.$$

Phase conventions

Clebsch–Gordan coefficients depend on phase conventions for basis states. Multiplying a basis vector by an overall phase changes some coefficients, but it does not change probabilities or physical predictions. Standard tables usually use the Condon–Shortley convention, in which the highest-weight coefficient is positive and the ladder coefficients are real and positive. This convention is also the one used in this chapter.

The phase issue is harmless if one uses one convention consistently. Difficulties arise only when coefficients from different sources are mixed without checking conventions. In atomic physics and spectroscopy, this is one reason it is important to state conventions when presenting detailed angular momentum calculations.

Physical role in selection rules

Clebsch–Gordan coefficients do more than rewrite states. They tell us which combinations of angular momenta can participate in a process. Suppose an atom emits or absorbs a photon carrying angular momentum one. The atomic angular momentum before and after the transition must combine with the photon's angular momentum in a way allowed by the triangle rule and by magnetic quantum number conservation. Many transitions are forbidden simply because the required coefficient is zero.

A full treatment uses tensor operators and the Wigner–Eckart theorem, but the basic idea is already visible here. Rotational symmetry separates a matrix element into a geometric angular part and a reduced dynamical part. Clebsch–Gordan coefficients contain the geometric part. The dynamics determines how strong an allowed transition is, but the angular momentum algebra determines whether it is allowed at all.

A useful mental picture

When reading a coupled state, it is helpful to ask two questions. First, what value of total m is required? This immediately selects the product states that can appear. Second, what symmetry or orthogonality condition distinguishes states with the same m? For two spin-$\frac{1}{2}$ particles with $m = 0$, the allowed product states are $|+\rangle\,|-\rangle$ and $|-\rangle\,|+\rangle$. Their symmetric combination belongs to the triplet, while their antisymmetric combination belongs to the singlet. More complicated examples use the same logic, but with larger subspaces.

A step-by-step workflow

For hand calculations, the following workflow is often effective. First identify j_1 and j_2, then list all product states grouped by total $m = m_1 + m_2$. Second, identify the highest m state; it must belong to the largest possible j. Third, apply J_- repeatedly to generate that whole multiplet.

Fourth, in each m-subspace, construct states orthogonal to the ones already assigned. These become the highest states of the next multiplets. Fifth, normalize and fix phases consistently.

This process is not always the fastest computational method, but it is pedagogically valuable because every step has a physical meaning. The highest state is unique because no other product state has that maximum projection. Lowering preserves total j. Orthogonality ensures distinct total-j multiplets. Normalization turns the expansion coefficients into probability amplitudes.

Interpreting zeros

Zero Clebsch–Gordan coefficients are as important as nonzero ones. A coefficient may vanish because $m \neq m_1 + m_2$, because the triangle rule fails, or because orthogonality forces a cancellation. In physical problems, these zeros often become selection rules. If a transition amplitude contains an angular factor equal to a zero Clebsch–Gordan coefficient, that transition is forbidden by angular momentum conservation or rotational symmetry.

This is why tables of coefficients are more than numerical references. Their pattern of zeros and signs reveals the structure of allowed couplings. In spectroscopy, scattering, and nuclear decay, much of the qualitative physics can be inferred by asking which angular momenta can combine and which coefficients vanish.

Entanglement in coupled states

Coupled states are often entangled with respect to the subsystem decomposition. The triplet states $|1,1\rangle$ and $|1,-1\rangle$ are simple product states, but $|1,0\rangle$ and $|0,0\rangle$ are not. They cannot be written as a state of particle 1 times a state of particle 2. Angular momentum coupling therefore naturally produces entanglement, even in systems as simple as two spin-$\frac{1}{2}$ particles.

This entanglement is not an extra assumption. It follows because states of definite total angular momentum are superpositions of product states. The coupled basis is adapted to symmetry; the product basis is adapted to subsystems. Quantum mechanics allows these two natural descriptions to be related by nontrivial superposition.

Example pattern for l plus spin-$\frac{1}{2}$

Although full tables for $l \otimes \frac{1}{2}$ are not needed here, the pattern is worth understanding. For an electron with orbital angular momentum l and spin $s = \frac{1}{2}$, the total angular momentum is either

$$j = l + \frac{1}{2} \quad \text{or} \quad j = l - \frac{1}{2}$$

for $l > 0$. A coupled state with quantum numbers j, m_j is a linear combination of at most two uncoupled product states:

$$|l, m_j - \tfrac{1}{2}\rangle\,|\tfrac{1}{2}, \tfrac{1}{2}\rangle\,, \qquad |l, m_j + \tfrac{1}{2}\rangle\,|\tfrac{1}{2}, -\tfrac{1}{2}\rangle\,,$$

provided the indicated m_l values lie in the allowed range. The Clebsch–Gordan coefficients give the weights of these two alternatives.

This two-term structure is one reason l-s coupling for a single electron is manageable. The total projection m_j fixes the sum $m_l + m_s$, and since m_s has only two possible values, only two orbital projections can contribute. The resulting coupled states are the spinor spherical harmonics used in more advanced treatments of the hydrogen atom and central potentials with spin.

Why tables are still useful

Even though coefficients can be derived, tables remain useful because angular momentum algebra appears repeatedly in calculations. A long atomic calculation may contain many angular momentum couplings, and deriving each coefficient from scratch would obscure the physics. Tables and software are efficient tools, but they should be used with understanding. One should always check the selection rule $m = m_1 + m_2$, the triangle rule, and the phase convention.

For learning, small examples such as two spin-$\frac{1}{2}$ particles are more valuable than large tables. They show the essential features: product states, coupled states, symmetry under exchange, normalization, and relative signs. Once these features are clear, larger tables become extensions of the same logic rather than mysterious arrays of square roots.

1.10 Angular Momentum and Rotation

Angular momentum is the generator of rotations. This statement is one of the most important links between symmetry and quantum mechanics. If a system is rotated by an angle θ about a unit vector $\hat{\mathbf{n}}$, the corresponding rotation operator is

$$R(\hat{\mathbf{n}}, \theta) = \exp\left(-\frac{\mathrm{i}}{\hbar}\theta\, \hat{\mathbf{n}} \cdot \mathbf{J}\right). \tag{1.100}$$

The operator $\mathbf{J}$ is the angular momentum that generates the rotation. For spatial wavefunctions it is orbital angular momentum $\mathbf{L}$. For spin states it is spin angular momentum $\mathbf{S}$. For a system with both orbital and spin degrees of freedom it is total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

For an infinitesimal rotation $\delta\theta$, expand the exponential:

$$R(\hat{\mathbf{n}}, \delta\theta) = 1 - \frac{\mathrm{i}}{\hbar}\delta\theta\, \hat{\mathbf{n}} \cdot \mathbf{J} + O(\delta\theta^2). \tag{1.101}$$

Thus a state transforms as

$$|\psi'\rangle = \left(1 - \frac{\mathrm{i}}{\hbar}\delta\theta\, \hat{\mathbf{n}} \cdot \mathbf{J}\right)|\psi\rangle\,. \tag{1.102}$$

This is analogous to translations, where linear momentum generates spatial displacements. Linear momentum is tied to translational symmetry; angular momentum is tied to rotational symmetry.

For a rotation about the z-axis,

$$R_z(\phi) = \exp\left(-\frac{\mathrm{i}}{\hbar}\phi J_z\right). \tag{1.103}$$

If $|j, m\rangle$ is an eigenstate of J_z, then

$$R_z(\phi)\,|j, m\rangle = \mathrm{e}^{-\mathrm{i}m\phi}\,|j, m\rangle\,. \tag{1.104}$$

Thus a state of definite m acquires a phase under rotation about the quantization axis. For orbital angular momentum this is consistent with the azimuthal dependence $Y_l^m(\theta, \phi) \propto \mathrm{e}^{\mathrm{i}m\phi}$: rotating the state and shifting the coordinate argument are inverse operations, which accounts for sign conventions in active and passive viewpoints.

An active rotation changes the physical state while keeping the coordinate axes fixed. A passive rotation changes the coordinate axes used to describe the same physical state. The two descriptions are closely related but have opposite signs in the rotation angle. Most quantum-mechanical formulas use active transformations for states:

$$|\psi\rangle \mapsto R\,|\psi\rangle\,.$$

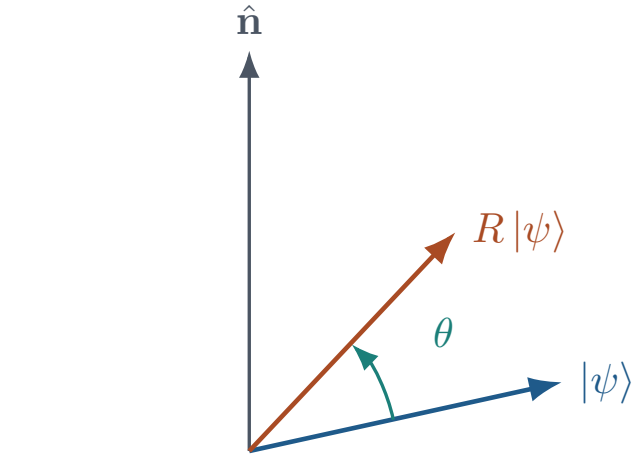

Figure 1.12: Angular momentum generates rotations. A finite rotation about $\hat{\mathbf{n}}$ is represented by the exponential of $-\mathrm{i}\theta\,\hat{\mathbf{n}}\cdot\mathbf{J}/\hbar$.

Observables transform as

$$A \mapsto A' = RAR^\dagger \tag{1.105}$$

if one rotates the apparatus or the observable. Equivalently, expectation values remain consistent:

$$\langle\psi'|\,A'\,|\psi'\rangle = \langle\psi|\,A\,|\psi\rangle$$

when state and observable are transformed together in the corresponding way.

Rotational invariance of a Hamiltonian means that rotating the system does not change the Hamiltonian:

$$RHR^\dagger = H \tag{1.106}$$

for all rotations R. For infinitesimal rotations this implies

$$[H, \hat{\mathbf{n}}\cdot\mathbf{J}] = 0 \tag{1.107}$$

for every direction $\hat{\mathbf{n}}$, and therefore

$$[H, J_x] = [H, J_y] = [H, J_z] = 0. \tag{1.108}$$

If H has no explicit time dependence, the Heisenberg equation gives

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}\,[H, J_i] = 0.$$

Thus rotational symmetry implies conservation of angular momentum.

This is the quantum version of the Noether idea: continuous symmetries lead to conserved quantities. In classical mechanics rotational invariance gives conservation of angular momentum. In quantum mechanics the same principle appears through commutators. A conserved observable commutes with the Hamiltonian; a symmetry transformation is generated by the corresponding conserved quantity.

The rotation group in ordinary three-dimensional space is SO(3). Spinors require a closely related group, SU(2), which double-covers SO(3). The phrase "double-covers" means that two elements of SU(2), differing by an overall sign, correspond to the same physical rotation in SO(3). This is why a spin-$\frac{1}{2}$ spinor changes sign under a 2π rotation. Observable probabilities are unchanged, but interference experiments can detect the phase structure.

The main lesson is that angular momentum is not only a set of measurable components. It is the infinitesimal engine of rotations. Once this is understood, the algebra

$$[J_i, J_j] = \mathrm{i}\hbar\epsilon_{ijk}J_k$$

is seen as the algebra of rotations themselves. This is why the same mathematics applies to orbitals, spinors, rigid rotors, and total atomic angular momentum.

Rotations of orbital wavefunctions

For a spinless particle, rotations act on the spatial wavefunction. An active rotation of the state by R may be written in coordinate representation as

$$\psi'(\mathbf{r}) = \psi(R^{-1}\mathbf{r}).$$

The inverse appears because the new amplitude at point $\mathbf{r}$ is the old amplitude at the point that rotates into $\mathbf{r}$. For an infinitesimal rotation about z,

$$x' = x + \delta\phi\, y, \qquad y' = y - \delta\phi\, x$$

in the inverse coordinate argument. Expanding the wavefunction to first order gives

$$\psi' = \psi - \delta\phi \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \psi = \left(1 - \frac{\mathrm{i}}{\hbar}\delta\phi\, L_z \right) \psi.$$

This explicitly identifies L_z as the generator of rotations about the z-axis.

The same reasoning works for rotations about x and y. The differential expressions for L_x, L_y, L_z are exactly the infinitesimal rotation generators acting on functions of position. Thus the operator definition $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is not an isolated quantization rule; it is the representation of spatial rotations on wavefunctions.

Rotational invariance and degeneracy

If $[H, \mathbf{J}] = 0$, then a rotated energy eigenstate has the same energy. Suppose

$$H\,|\psi\rangle = E\,|\psi\rangle\,.$$

If R commutes with H, then

$$H(R\,|\psi\rangle) = R(H\,|\psi\rangle) = E(R\,|\psi\rangle).$$

Therefore rotations produce new states with the same energy. If the original state is part of a nontrivial angular momentum multiplet, this gives degeneracy among different m values. The degeneracy is not accidental; it is enforced by symmetry.

When a perturbation breaks rotational symmetry, the degeneracy may be lifted. A magnetic field along z, for example, preserves rotations about z but not about x or y. Then J_z may remain conserved while J^2 or the full vector $\mathbf{J}$ no longer has the same status. This symmetry reduction is the conceptual basis of many splitting effects.

Finite rotations and noncommutativity

Finite rotations about different axes do not generally commute. In quantum mechanics this is encoded in the noncommuting generators inside exponentials. If A and B do not commute, then

$$\mathrm{e}^{A}\mathrm{e}^{B} \neq \mathrm{e}^{A+B}$$

in general. The Baker–Campbell–Hausdorff formula begins

$$\mathrm{e}^{A}\mathrm{e}^{B} = \exp\left(A + B + \frac{1}{2}\,[A,B] + \cdots\right).$$

For rotations, the commutator term is another rotation generator. This is the exponential version of the angular momentum algebra.

For spin-$\frac{1}{2}$, the finite rotation formula has the closed form

$$U(\hat{\mathbf{n}},\theta) = \cos\frac{\theta}{2}I - \mathrm{i}\sin\frac{\theta}{2}\,\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}.$$

For orbital angular momentum, finite rotations are represented in each l subspace by $(2l+1)$-dimensional matrices known as Wigner D-matrices. The details of these matrices are beyond our present needs, but the principle is the same: exponentiate the angular momentum generators.

Diagnosing symmetry with commutators

In practice, one often decides whether angular momentum is conserved by inspecting commutators. If the Hamiltonian contains only scalar quantities under rotations, such as p^2, r, or $\mathbf{L}\cdot\mathbf{S}$, then it is rotationally invariant and commutes with the total angular momentum. If it contains a term such as z, L_z, or $\mathbf{B}\cdot\mathbf{S}$ with a fixed external vector $\mathbf{B}$, then full rotational symmetry is reduced or broken.

The important question is always: what is being rotated? If an external magnetic field is part of the physical environment and is held fixed, it selects a direction and breaks rotational symmetry of the particle Hamiltonian. If one rotates both the particle and the field together, the combined description may still be covariant, but the particle alone no longer has full rotational invariance. This distinction prevents many conceptual mistakes.

Good quantum numbers

A quantum number is called good when the corresponding operator commutes with the Hamiltonian and can be used to label stationary states. In a central potential without spin, l and m are good quantum numbers. In a spin-orbit coupled atom, l, s, j, m_j may be good, while m_l and m_s separately may not be. In a strong magnetic field, the situation may change again. Good quantum numbers are therefore not permanent names attached to a system; they depend on the Hamiltonian and its symmetries.

Rotational symmetry is especially useful because it often survives approximations. Even when the radial form of a central potential is complicated, the angular momentum labels remain valid as long as the potential depends only on r. This lets physicists classify states before solving detailed energy equations. The symmetry analysis narrows the problem.

Active and passive viewpoints revisited

The active-passive distinction can be summarized in one sentence: active rotations move the state; passive rotations move the coordinate description. Both viewpoints are valid, but signs in

formulas depend on which one is used. In active language, the state becomes $R|\psi\rangle$. In passive language, the coordinates are rotated and the same physical state is described by transformed components. Confusing the two viewpoints is a common source of sign errors in rotation formulas.

For most operator work in quantum mechanics, active transformations are cleaner. Observables and states transform by unitary operators, and invariance is expressed by commutators. When working with explicit wavefunctions, passive coordinate changes may feel more natural. The two pictures agree when used consistently.

Rotations as a test of physical laws

One way to understand a symmetry is to imagine performing an experiment in a rotated laboratory. If every apparatus, source, detector, and coordinate label is rotated together, the probabilities predicted by the theory should be unchanged in an isotropic world. This does not mean every state is unchanged by rotation. A p_z-like state rotated by ninety degrees becomes a p_x-like state. The statement is that the laws relating states, observables, and probabilities have the same form.

Quantum mechanics implements this idea with unitary operators. Rotations preserve inner products, and therefore preserve probabilities. The generator of those unitary rotations is angular momentum. This is why the exponential form of the rotation operator is so important: it is the mathematical mechanism by which a continuous spatial symmetry acts on Hilbert space.

This perspective also explains why conserved quantities are tied to symmetries. If time evolution is governed by a Hamiltonian that looks the same after rotation, then rotating a state before or after time evolution gives the same result. In operator language, the rotation operator commutes with the Hamiltonian, and the infinitesimal version says that angular momentum commutes with the Hamiltonian. Conservation is therefore not an extra rule; it is the dynamical consequence of symmetry.

Scalar, vector, and tensor operators

Operators themselves can be classified by how they transform under rotations. A scalar operator is unchanged by rotations; examples include r, p^2, L^2, and $\mathbf{L}\cdot\mathbf{S}$. A vector operator has three components that mix like x, y, z under rotations; examples include $\mathbf{r}$, $\mathbf{p}$, $\mathbf{L}$, and $\mathbf{S}$. Higher-rank tensor operators have more components and more elaborate transformation laws.

This classification is the deeper origin of selection rules. If an interaction is a scalar, it cannot change total angular momentum in arbitrary ways. If an interaction is a vector, it carries one unit of angular momentum and obeys different rules. The Clebsch–Gordan machinery is the bookkeeping system for these transformation properties. Although a full tensor-operator treatment is beyond this chapter, the essential idea is already present: how an operator transforms determines which matrix elements can be nonzero.

1.11 Motion in a Centrally Symmetric Field

A central potential is a potential that depends only on the distance from a fixed origin:

$$V = V(r), \qquad r = \sqrt{x^2 + y^2 + z^2}. \tag{1.109}$$

The Hamiltonian for a particle of mass m in such a potential is

$$H = \frac{\mathbf{p}^2}{2m} + V(r) = -\frac{\hbar^2}{2m}\nabla^2 + V(r). \tag{1.110}$$

Central potentials are rotationally symmetric. Rotating the system changes the direction of $\mathbf{r}$ but not its length, so $V(r)$ is unchanged. This symmetry is the reason angular momentum is conserved.

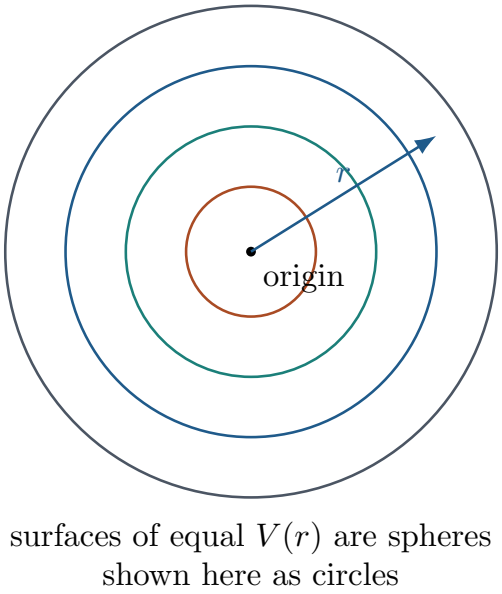

Figure 1.13: A central potential depends only on the radial distance r. Its surfaces of constant potential are spheres, shown here as circles in a plane.

Because the kinetic energy $\mathbf{p}^2/2m$ is also rotationally invariant, the full Hamiltonian commutes with the generators of rotations:

$$[H, L_x] = [H, L_y] = [H, L_z] = 0. \tag{1.111}$$

It follows that

$$\left[H, L^2\right] = 0, \qquad [H, L_z] = 0. \tag{1.112}$$

Therefore stationary states may be chosen to be simultaneous eigenstates of H, L^2, and L_z. This is the mathematical origin of the quantum numbers l and m in central-field problems.

Spherical coordinates are adapted to the symmetry:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

The Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{1.113}$$

Using Eq. (1.12), this can be written as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}. \tag{1.114}$$

The time-independent Schrodinger equation is

$$H\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{1.115}$$

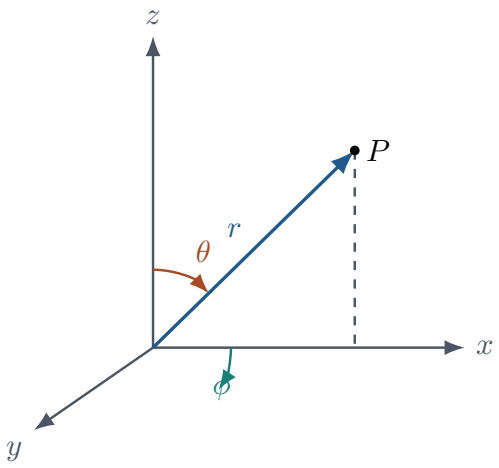

Figure 1.14: Spherical coordinates separate the radial distance r from the angular variables θ and ϕ, making them natural for central potentials.

Because H, L^2, and L_z commute, we try a separated form

$$\psi(r,\theta,\phi) = R(r)Y_l^m(\theta,\phi). \tag{1.116}$$

The angular functions satisfy

$$L^2 Y_l^m(\theta,\phi) = \hbar^2 l(l+1)Y_l^m(\theta,\phi), \tag{1.117}$$
$$L_z Y_l^m(\theta,\phi) = m\hbar Y_l^m(\theta,\phi). \tag{1.118}$$

They are the spherical harmonics. Their quantum numbers have the same meaning as before: l determines the total orbital angular momentum and m determines its projection along z.

Substituting Eq. (1.116) into the Schrodinger equation and using Eq. (1.114) gives the radial equation

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left[V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}\right]R = ER. \tag{1.119}$$

The term

$$\frac{\hbar^2 l(l+1)}{2mr^2} \tag{1.120}$$

is called the centrifugal barrier. It is repulsive and becomes large near the origin for $l > 0$. Although its name is borrowed from classical mechanics, it is a quantum term arising from angular kinetic energy.

It is often cleaner to define the reduced radial function

$$u(r) = rR(r).$$

Then the radial equation becomes a one-dimensional Schrodinger-like equation:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + V_{\text{eff}}(r)u(r) = Eu(r), \tag{1.121}$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}. \tag{1.122}$$

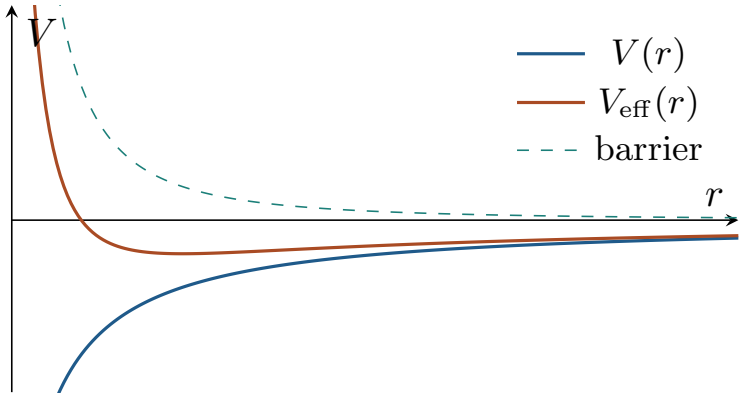

Figure 1.15: For $l > 0$, the effective potential contains a positive centrifugal barrier proportional to $l(l+1)/r^2$.

This effective potential is a powerful way to understand the radial motion. The original potential governs attraction or confinement, while the centrifugal term suppresses probability near the origin when $l \neq 0$.

The angular and radial parts of the wavefunction carry different physical information. The spherical harmonic Y_l^m determines the angular pattern: nodes, lobes, and orientation relative to the chosen quantization axis. The radial function determines how probability is distributed with distance from the origin. The probability of finding the particle between r and $r + \mathrm{d}r$ is proportional to

$$r^2|R(r)|^2\,\mathrm{d}r = |u(r)|^2\,\mathrm{d}r,$$

after integrating over angles.

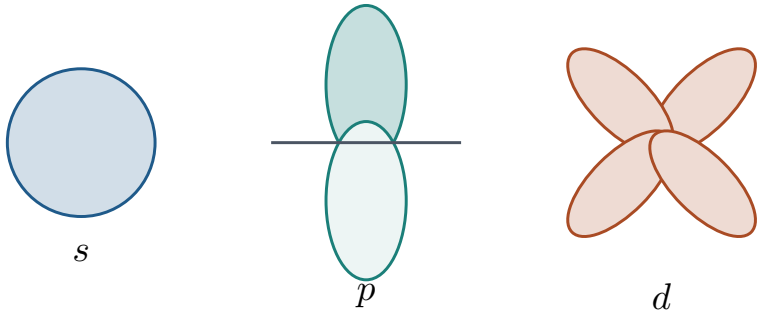

Figure 1.16: Schematic angular patterns associated with s, p, and d harmonics. The drawings indicate nodal structure and symmetry rather than exact probability densities.

Rotational symmetry also explains degeneracy. If the Hamiltonian contains only $V(r)$, it commutes with all components of $\mathbf{L}$, and energy cannot depend on the orientation quantum number m. States with the same radial quantum numbers and the same l, but different m, are degenerate unless an external field or additional interaction selects a direction. This is why magnetic fields split m levels: the field breaks full rotational symmetry down to rotations about the field axis.

The central-field problem is therefore the meeting point of operator algebra and wave mechanics. The angular momentum algebra predicts the angular eigenvalues and the allowed

m values. Separation of variables supplies the spherical harmonics and the radial equation. Together they reduce a three-dimensional quantum problem with spherical symmetry to a family of one-dimensional radial problems labelled by l.

Boundary conditions and radial normalization

The radial equation must be supplemented by physical boundary conditions. Bound-state wavefunctions must be normalizable:

$$\int |\psi|^2 \, \mathrm{d}^3 r = 1.$$

Using the separated form and normalized spherical harmonics, this becomes

$$\int_0^\infty |R(r)|^2 r^2 \, \mathrm{d}r = 1,$$

or equivalently

$$\int_0^\infty |u(r)|^2 \, \mathrm{d}r = 1$$

for $u(r) = rR(r)$. The reduced radial function is useful because it obeys a one-dimensional-looking equation with the usual measure $\mathrm{d}r$.

Regularity at the origin is another condition. For $l > 0$, the centrifugal barrier strongly suppresses the wavefunction near $r = 0$. Typically the regular radial solution behaves as

$$R(r) \sim r^l$$

near the origin. Singular solutions are rejected because they fail to represent acceptable physical states. At large r, bound-state radial functions must decay sufficiently rapidly for normalization.

Angular nodes and radial nodes

Central-field wavefunctions have two kinds of nodes. Angular nodes come from spherical harmonics and are determined by l and m. Radial nodes come from $R(r)$ and depend on the radial quantum structure of the problem. In hydrogen, the number of radial nodes is

$$n - l - 1.$$

Although this formula is specific to the Coulomb problem, the distinction between angular and radial nodes is general. Angular momentum controls the angular part; the radial potential controls the radial part.

This separation is one reason orbital shapes can be discussed in a systematic way. The s-states are spherically symmetric because Y_0^0 is constant. The p-states have one angular node. The d-states have more elaborate angular nodal structure. The radial function may add spherical nodal shells, but it does not change the angular momentum quantum number.

Degeneracy and broken symmetry

For a central potential, energy eigenvalues do not depend on m. This follows from rotational symmetry alone. Whether energy depends on l depends on the specific form of $V(r)$. In a generic central potential, different l values are not degenerate because the centrifugal barrier changes the radial equation. The Coulomb potential and the three-dimensional harmonic oscillator have additional symmetries that produce larger degeneracies.

If a weak external field is added, the symmetry changes. A magnetic field along z introduces a preferred axis. The Hamiltonian may still commute with L_z, but it no longer commutes with L_x and L_y. The degeneracy among different m values can then be lifted. This is the central mechanism behind magnetic splitting of atomic levels.

Classical correspondence

The centrifugal term in the radial equation has a useful classical analogue. A classical particle with angular momentum L moving in a central potential has radial energy

$$E = \frac{p_r^2}{2m} + V(r) + \frac{L^2}{2mr^2}.$$

Quantum mechanically L^2 is replaced by $\hbar^2 l(l+1)$. This replacement is not simply $l^2\hbar^2$; the $l(l+1)$ factor is a genuine quantum result. In the large-l limit, $l(l+1) \approx l^2 + l$, and the difference between l and $\sqrt{l(l+1)}$ becomes relatively small. This is one way the classical picture emerges from the quantum one at large angular momentum.

Solving a central-field problem in practice

The practical solution of a central-field problem follows a standard sequence. First, identify that V depends only on r. This immediately tells us that H, L^2, and L_z commute. Second, write the wavefunction as $R(r)Y_l^m(\theta, \phi)$. Third, use the known angular eigenvalues of L^2 to reduce the Schrodinger equation to a radial equation. Fourth, solve the radial equation with the appropriate boundary conditions. The angular part is universal; the radial part depends on the specific potential.

This division of labor is powerful. The same spherical harmonics appear for the hydrogen atom, a finite spherical well, an isotropic harmonic oscillator, and many model nuclear potentials. What changes from one system to another is the radial equation and therefore the allowed energies and radial functions. Angular momentum theory supplies a reusable angular framework.

The meaning of m-degeneracy

The degeneracy with respect to m can be understood without solving the radial equation. If a state with a particular m has energy E, then rotating that state produces another state with the same energy. A rotationally invariant Hamiltonian cannot assign different energies to different orientations of the same angular pattern. The $2l + 1$ values of m are different basis states within this orientation multiplet.

However, the choice of m-basis is not unique. A real p_x orbital, for example, is a linear combination of $m = 1$ and $m = -1$ states. In a perfectly central potential, it has the same energy as p_y, p_z, and the complex m-basis states. Once an external field or molecular environment selects axes, some combinations may become more natural than others.

Centrifugal barrier and penetration

The centrifugal barrier has important physical consequences. For $l = 0$, there is no angular barrier, so the wavefunction can have significant probability density near the origin. For $l > 0$, the barrier suppresses the radial wavefunction at small r. In atomic physics this affects penetration: s-electrons can penetrate close to the nucleus more strongly than p- or d-electrons. In multi-electron atoms this contributes to the ordering of subshell energies.

The same idea appears in scattering. At low energies, high-l partial waves are suppressed because the centrifugal barrier prevents the particle from reaching the interaction region. The $l = 0$ partial wave, called the s-wave, often dominates low-energy scattering. Thus angular momentum is not only a bound-state label; it also organizes continuum motion.

Partial waves

Any reasonable wavefunction can be expanded in spherical harmonics:

$$\psi(r, \theta, \phi) = \sum_{l,m} R_{lm}(r) Y_l^m(\theta, \phi).$$

In a central potential, different l, m channels do not mix. This is the partial-wave decomposition. It is one of the main tools in scattering theory, where an incoming wave is decomposed into angular momentum channels. Each channel sees a different centrifugal barrier and acquires a different phase shift. Even though scattering theory is beyond the scope of this chapter, the same angular momentum machinery is already present in the central-field separation.

Central potentials beyond the ideal case

Real physical systems are rarely perfectly central, but central potentials are still the starting point for many approximations. In atoms, the average field felt by an electron may be approximated as central even though the exact electron-electron interaction is not simply a function of one electron's radius. In nuclei, shell models often begin with an average spherical potential plus corrections. In molecules, a central approximation may describe an isolated atom before bonding or crystal fields break the symmetry.

When the central symmetry is only approximate, l and m may stop being exact quantum numbers, but they can remain useful approximate labels. A weak noncentral perturbation mixes states with different angular momentum according to its transformation properties. For example, an electric field along z can mix states of opposite parity and different l, leading to Stark shifts. The unperturbed central-field basis is still the natural language for calculating those shifts.

Angular momentum and parity

Spherical harmonics have definite parity:

$$Y_l^m(\pi - \theta, \phi + \pi) = (-1)^l Y_l^m(\theta, \phi).$$

Thus an orbital state with angular momentum l has parity $(-1)^l$ if the radial part depends only on r. This adds another useful label in central potentials. Even l states have even parity; odd l states have odd parity. Selection rules often involve both angular momentum and parity. For example, electric dipole transitions change parity, which is consistent with the rule $\Delta l = \pm 1$ in hydrogenic systems.

Parity is not the main subject of this chapter, but it works naturally with angular momentum because both are tied to spatial symmetry. Together they provide a compact classification of states in central potentials.

1.12 Hydrogen Atom

The hydrogen atom is the central-field problem of greatest historical and conceptual importance. In the simplest nonrelativistic treatment, the proton is taken to be fixed at the origin and the

electron moves in the Coulomb potential

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}. \tag{1.123}$$

More accurately, one replaces the electron mass by the reduced mass μ of the electron-proton system. The Hamiltonian is

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{4\pi\epsilon_0 r}. \tag{1.124}$$

Because the potential depends only on r, the hydrogen atom is rotationally symmetric. Thus H, L^2, and L_z can be diagonalized simultaneously.

The stationary wavefunctions separate as

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi). \tag{1.125}$$

The angular part is exactly the same spherical harmonic structure as in any central potential. The radial part is determined by the Coulomb potential and by normalizability. The allowed quantum numbers are

$$n = 1, 2, 3, \ldots, \qquad l = 0, 1, \ldots, n-1, \qquad m = -l, -l+1, \ldots, l. \tag{1.126}$$

The principal quantum number n controls the energy and the overall radial scale. The orbital quantum number l controls the angular momentum and contributes to the radial shape through the centrifugal barrier. The magnetic quantum number m controls the projection of orbital angular momentum along the chosen axis.

The bound-state energies are

$$E_n = -\frac{13.6\,\mathrm{eV}}{n^2}, \tag{1.127}$$

when the reduced-mass correction is neglected. The energy depends only on n, not on l or m. The absence of m-dependence follows from rotational symmetry. The absence of l-dependence is more special: it is a hidden degeneracy of the Coulomb potential associated with an additional conserved quantity, the Runge–Lenz vector. For generic central potentials, energies depend on l as well as on radial quantum numbers.

For a fixed n, the number of orbital states is

$$\sum_{l=0}^{n-1}(2l+1) = n^2. \tag{1.128}$$

Including electron spin doubles this to $2n^2$, before fine structure, hyperfine structure, external fields, or the Pauli principle in multi-electron atoms are considered.

The spectroscopic labels for orbital angular momentum are

$$l = 0 \to s, \qquad l = 1 \to p, \qquad l = 2 \to d, \qquad l = 3 \to f,$$

with later values continuing alphabetically in standard notation. Thus a state with $n = 2, l = 1$ is a $2p$ state, while a state with $n = 3, l = 2$ is a $3d$ state. The label s means zero orbital angular momentum; p, d, and f correspond to increasing angular structure.

The angular momentum magnitude in a hydrogenic orbital is

$$|\mathbf{L}| = \sqrt{l(l+1)}\hbar.$$

For s-states, $l = 0$, so the orbital angular momentum is zero and the angular wavefunction is spherically symmetric. For p-states, $l = 1$, so $|\mathbf{L}| = \sqrt{2}\hbar$, and there are three possible m

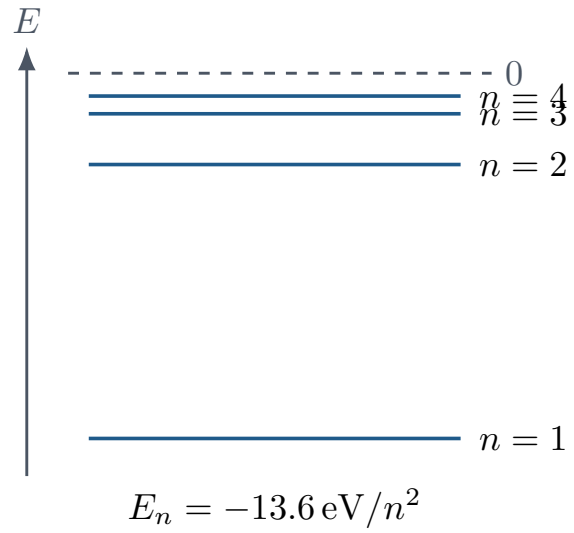

Figure 1.17: Nonrelativistic hydrogen energy levels depend only on n. The levels approach $E = 0$, the ionization threshold, as $n \to \infty$.

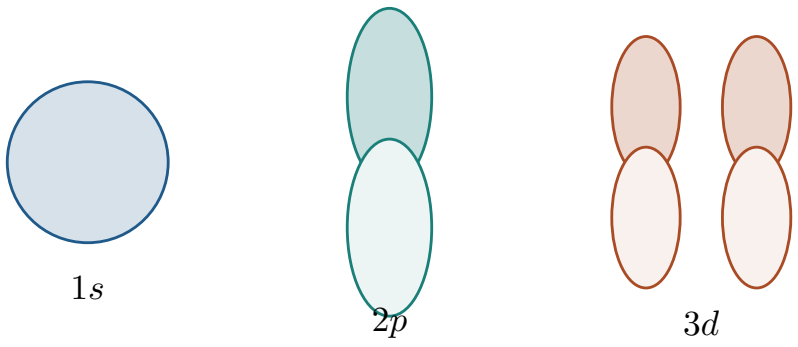

Figure 1.18: Schematic hydrogen orbital shapes. The exact orbitals are probability amplitudes with radial and angular structure; the sketches emphasize the angular symmetry.

values. For d-states, $l = 2$, there are five possible m values. These multiplicities are the familiar degeneracies of orbital sublevels.

The angular functions also determine nodal surfaces. An s-orbital has no angular node. A p-orbital has one angular nodal plane. A d-orbital has two angular nodes in suitable real combinations. The complex spherical harmonics Y_l^m are eigenstates of L_z, but chemists often use real linear combinations because they display spatial orientation more directly. Both descriptions span the same l subspace.

Electron spin must be included for a complete description. The electron has

$$s = \frac{1}{2}, \qquad m_s = \pm\frac{1}{2}.$$

The total angular momentum is

$$\mathbf{J} = \mathbf{L} + \mathbf{S}. \tag{1.129}$$

For a given l, the allowed total j values are

$$j = l + \frac{1}{2}, \qquad j = l - \frac{1}{2}$$

with the second value absent for $l = 0$. The corresponding states are labelled by n, l, j, m_j when spin-orbit effects are important.

Spin-orbit coupling is a relativistic correction that couples $\mathbf{L}$ and $\mathbf{S}$. Its detailed derivation requires magnetic fields in the electron rest frame and Thomas precession, but its angular structure is simple:

$$H_{\mathrm{SO}} \propto \mathbf{L} \cdot \mathbf{S}.$$

Using

$$\mathbf{L} \cdot \mathbf{S} = \frac{1}{2}(J^2 - L^2 - S^2),$$

we see that states of good J^2, L^2, S^2 diagonalize the angular part of the spin-orbit interaction. This is a direct application of angular momentum addition and Clebsch–Gordan coefficients.

Fine structure removes part of the nonrelativistic degeneracy. Relativistic kinetic-energy corrections, spin-orbit coupling, and the Darwin term shift the levels so that the energy depends on n and j in the Dirac theory, rather than only on n. Hyperfine structure, caused by coupling between electron and proton magnetic moments, splits levels further. External magnetic and electric fields produce Zeeman and Stark effects. Each refinement uses angular momentum as the organizing language.

Hydrogen is therefore not merely an application after the theory is finished. It is one of the strongest reasons the theory is built as it is. The quantum numbers n, l, m emerge from separation of variables and angular momentum algebra. The shapes of orbitals are controlled by spherical harmonics. Degeneracies reflect symmetry. Spin and total angular momentum explain fine structure and selection rules. A large fraction of atomic physics is contained in the statement that hydrogen is a rotationally symmetric Coulomb problem with a spin-$\frac{1}{2}$ electron.

Hydrogen quantum numbers

For nonrelativistic hydrogen,

$$\psi_{nlm} = R_{nl}(r)Y_l^m(\theta, \phi), \qquad E_n = -\frac{13.6\,\mathrm{eV}}{n^2},$$

with

$$n = 1, 2, \ldots, \qquad l = 0, \ldots, n-1, \qquad m = -l, \ldots, l.$$

The orbital degeneracy of the n-th level is n^2, or $2n^2$ when electron spin is included.

Radial scale and the Bohr radius

The natural length scale of hydrogen is the Bohr radius,

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2},$$

which is approximately 0.529 angstrom when the reduced mass is replaced by the electron mass. The radial wavefunctions are built from exponentials, powers of r, and associated Laguerre polynomials. Their detailed form is not needed for the angular momentum theory, but their structure reinforces the separation:

$$R_{nl}(r) \quad \text{depends on } n, l, \qquad Y_l^m(\theta, \phi) \quad \text{depends on } l, m.$$

The same l appears in both parts because angular momentum affects the radial equation through the centrifugal barrier.

The ground state has $n = 1, l = 0, m = 0$. Its angular part is constant, and its radial part decays exponentially:

$$R_{10}(r) \propto \mathrm{e}^{-r/a_0}.$$

This state is spherically symmetric. The first excited shell $n = 2$ contains $2s$ and $2p$ orbitals. The $2s$ orbital is still spherically symmetric but has a radial node, while the $2p$ orbitals have angular structure and one unit of orbital angular momentum.

Why $l < n$

The restriction $l = 0, 1, \ldots, n-1$ comes from the normalizable solutions of the Coulomb radial equation. Roughly speaking, larger l increases the centrifugal barrier and requires more radial excitation to support a bound state. The principal quantum number n counts the total excitation in a way that includes both radial and angular structure. The number of radial nodes is

$$n - l - 1,$$

which must be a nonnegative integer. Therefore $l \leq n - 1$.

This relation explains the shell structure. The $n = 1$ shell has only s-states. The $n = 2$ shell has s and p-states. The $n = 3$ shell has s, p, d-states. In multi-electron atoms the energies depend strongly on l because shielding and electron-electron interactions destroy the pure Coulomb degeneracy, but the allowed angular momentum labels remain the same.

Complex and real orbitals

The functions Y_l^m with definite m are generally complex for $m \neq 0$. They are the natural eigenfunctions of L_z. In chemistry, however, one often draws real orbitals such as p_x, p_y, p_z or real d-orbitals. These are linear combinations of the complex spherical harmonics with the same l. For example, the p_z orbital corresponds to Y_1^0, while p_x and p_y are real combinations of Y_1^1 and Y_1^{-1}.

This does not contradict angular momentum theory. The real orbitals are usually not eigenstates of L_z, but they still lie in the $l = 1$ subspace and are eigenstates of L^2. Choosing complex or real orbitals is a choice of basis within a degenerate subspace. If a magnetic field defines the z-axis, the complex m-basis becomes physically convenient. If molecular geometry defines Cartesian directions, real orbitals may be more visually useful.

Spin, fine structure, and the hierarchy of approximations

The simplest hydrogen Hamiltonian ignores spin and relativity. At that level the energy depends only on n. Adding spin doubles the number of states but does not by itself change the energy unless a spin-dependent interaction is present. Relativistic corrections then introduce fine structure, where j becomes an important label. Hyperfine interactions introduce the proton spin and total atomic angular momentum including nuclear spin. External fields introduce additional splittings.

The hierarchy can be summarized as follows. The largest structure is the Coulomb energy labelled by n. The next refinements distinguish states by j, l, and spin couplings. External fields may distinguish m_j. Each stage is governed by angular momentum: which operators commute with the Hamiltonian, which quantum numbers remain good, and which degeneracies survive.

Angular momentum and spectral lines

When hydrogen emits or absorbs light, angular momentum constrains the transition. Electric dipole transitions obey selection rules such as

$$\Delta l = \pm 1, \qquad \Delta m = 0, \pm 1,$$

with more detailed rules when spin and total angular momentum are included. These rules arise because the dipole operator transforms like a vector under rotations. The transition amplitude contains angular integrals over spherical harmonics, and many of these integrals vanish by angular momentum algebra. Thus even spectroscopy is organized by the same ideas developed in this chapter.

Hydrogen as a template for atomic structure

Hydrogen is special because it has only one electron, but the angular momentum language developed from hydrogen survives in multi-electron atoms. Electrons are still assigned orbital labels $s, p, d, f, \ldots$, and spin remains $s = \frac{1}{2}$. The difference is that electron-electron repulsion and screening change the energy ordering. In hydrogen, all states with the same n are degenerate before fine structure. In multi-electron atoms, states with different l but the same principal shell generally have different energies.

Despite this complication, angular momentum remains the organizing principle. Atomic terms, fine-structure levels, and selection rules are labelled by total orbital, spin, and total angular momentum quantum numbers. The detailed coupling scheme may vary: light atoms are often described by LS coupling, while heavier atoms may require jj coupling. Both schemes are applications of the same angular momentum addition theory.

Degeneracy hierarchy in hydrogen

The degeneracy of hydrogen has several layers. Rotational symmetry guarantees degeneracy in m. The Coulomb potential adds degeneracy among different l values with the same n. Spin doubles the degeneracy if spin-dependent interactions are ignored. Fine structure and hyperfine structure then remove parts of this degeneracy. External fields remove still more. Understanding which degeneracy is due to which symmetry is more important than memorizing the final splitting pattern.

For example, the $n = 2$ level contains $2s$ and $2p$ states. In the simplest Schrodinger theory they have the same energy. Fine structure and the Lamb shift modify this statement in real

hydrogen. The Lamb shift, a quantum electrodynamic effect, separates $2s_{1/2}$ and $2p_{1/2}$, showing that even when angular momentum gives the main classification, additional physics can refine the spectrum.

Orbital angular momentum and probability density

The phrase "orbital shape" can be misleading if taken too literally. The electron is not moving along the surface of a drawn lobe. The orbital is a wavefunction, and the plotted shapes usually represent surfaces of constant probability density or angular amplitude. The angular momentum quantum number l determines the symmetry of that distribution. The magnetic quantum number m, or a real linear combination of m-states, determines orientation features relative to chosen axes.

For s-states the probability density is spherically symmetric. For p-states the density has a nodal plane. For d-states the density has more complex nodal surfaces. These shapes matter because they influence bonding, transition strengths, and spatial overlap in atoms and molecules. Angular momentum therefore connects abstract operator algebra to visual and chemical intuition.

The role of reduced mass

In a more accurate treatment, the proton is not fixed. The electron and proton both move about their center of mass. The two-body problem separates into free center-of-mass motion and relative motion with the reduced mass

$$\mu = \frac{m_e m_p}{m_e + m_p}.$$

The angular momentum in the relative coordinate is the orbital angular momentum of the internal hydrogen state. Replacing m_e by μ slightly shifts the energy scale. The angular momentum classification is unchanged, but precision spectroscopy requires the reduced mass and many smaller corrections.

Hidden symmetry and the special Coulomb degeneracy

Rotational symmetry explains why energy does not depend on m, but it does not by itself explain why the nonrelativistic Coulomb energy does not depend on l. That larger degeneracy is special to the $1/r$ potential. Classically, the Kepler problem has a conserved Runge–Lenz vector that fixes the orientation of the elliptical orbit. Quantum mechanically, an analogous conserved operator enlarges the symmetry beyond ordinary rotations.

The practical consequence is that all orbital states with the same n have the same Schrodinger energy, even though they have different angular momentum and different radial shapes. A $3s$, $3p$, and $3d$ state are degenerate in the simplest theory. This is not true for a generic central potential. The distinction is important: m-degeneracy is a general result of rotational symmetry, while l-degeneracy in hydrogen is a special result of the Coulomb potential.

Preparing for more advanced hydrogen theory

The angular momentum framework developed here is exactly the framework used in more advanced treatments. In the Dirac equation for hydrogen, the electron spin is built in from the start, and states are labelled by total angular momentum j and m_j. The orbital quantum number l is still important, but it is tied to the spinor structure of the relativistic wavefunction.

Fine structure, spin-orbit coupling, and relativistic corrections all respect rotational symmetry, so total angular momentum remains the correct organizing label.

Quantum electrodynamics adds still finer corrections, such as the Lamb shift and anomalous magnetic moment. These effects require physics beyond the Schrodinger equation, yet their classification still uses angular momentum. This illustrates a general theme: even when the dynamics becomes more sophisticated, symmetry labels remain stable and powerful.

Why hydrogen belongs in an angular momentum chapter

Hydrogen is often taught as a separate exactly solvable problem, but it is best understood as a culmination of angular momentum theory. The separation into R_{nl} and Y_l^m uses L^2. The magnetic degeneracy uses rotational symmetry. The orbital labels use integer l. The inclusion of electron spin uses spin-$\frac{1}{2}$ representations. Fine structure uses angular momentum addition. Spectroscopic selection rules use Clebsch–Gordan logic.

Thus hydrogen is not merely an example after the formalism; it is a physical system in which nearly every part of the formalism becomes visible. A student who understands the angular momentum organization of hydrogen has acquired tools that transfer directly to atoms, molecules, nuclei, and quantum information systems.

A note on notation in spectroscopy

Spectroscopic notation compresses angular momentum information into compact labels. A one-electron orbital label such as $3d$ means $n = 3$ and $l = 2$. When spin-orbit coupling is included, one may write labels such as $3d_{5/2}$ or $3d_{3/2}$, where the subscript is j. The magnetic quantum number m_j is often not written unless a magnetic field or polarization selection rule makes it necessary.

In multi-electron atoms, term symbols such as ${}^{2S+1}L_J$ are used. Here S is total spin, L is total orbital angular momentum written as $S, P, D, F, \ldots$ for $L = 0, 1, 2, 3, \ldots$, and J is total angular momentum. The notation is compact but can be confusing because the letter S is used both for spin and for the $L = 0$ orbital letter in different positions. The context and typography matter.

The purpose of mentioning this notation is not to develop multi-electron spectroscopy, but to show how strongly angular momentum shapes the language of atomic physics. Spectral tables are not arbitrary lists of names. They are organized by the same quantum numbers and coupling rules developed here.

What survives when approximations change

A simple model may be improved in many ways: finite proton mass, relativistic kinetic energy, spin-orbit coupling, quantum electrodynamic shifts, hyperfine coupling, or external fields. Each improvement changes the Hamiltonian. The good quantum numbers may change as well. Yet the angular momentum framework remains the diagnostic tool for deciding what survives. If the correction is rotationally invariant, total angular momentum remains conserved. If the correction selects an axis, only the projection along that axis may remain conserved.

This is a general lesson about quantum numbers. They are not merely names of states; they are eigenvalues of operators that commute with the Hamiltonian under specified approximations. When the Hamiltonian changes, one must ask again which operators commute with it. Hydrogen is an ideal training ground for this habit because one can see the gradual refinement from n, l, m to n, l, j, m_j and beyond.

1.13 Worked Examples

The examples in this section collect the main techniques of the chapter. They are deliberately written with intermediate steps, because angular momentum calculations are often simple in principle but sensitive to signs, square-root factors, and basis conventions.

Example 1: Derive $[L_x, L_y] = \mathrm{i}\hbar L_z$

Problem. Starting from

$$L_x = yp_z - zp_y, \qquad L_y = zp_x - xp_z,$$

and the canonical commutators $[x_i, p_j] = \mathrm{i}\hbar\delta_{ij}$, derive $[L_x, L_y]$.
Solution. Expand the commutator:

$$\begin{aligned}[L_x, L_y] &= [yp_z - zp_y, zp_x - xp_z] \\ &= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]\,.\end{aligned}$$

The second and third terms vanish because they contain no conjugate coordinate-momentum pair in a nontrivial order. For the first term,

$$\begin{aligned}[yp_z, zp_x] &= yp_z zp_x - zp_x yp_z \\ &= y(zp_z - \mathrm{i}\hbar)p_x - zyp_x p_z \\ &= -\mathrm{i}\hbar\, yp_x.\end{aligned}$$

For the last term,

$$\begin{aligned}[zp_y, xp_z] &= zp_y xp_z - xp_z zp_y \\ &= zxp_y p_z - x(zp_z - \mathrm{i}\hbar)p_y \\ &= \mathrm{i}\hbar\, xp_y.\end{aligned}$$

Therefore

$$[L_x, L_y] = \mathrm{i}\hbar(xp_y - yp_x) = \mathrm{i}\hbar L_z.$$

Final answer

$$\boxed{[L_x, L_y] = \mathrm{i}\hbar L_z.}$$

Physical interpretation. The x- and y-components of angular momentum are incompatible observables. Their noncommutativity is measured by the third component, L_z.

Example 2: Show that $\left[L^2, L_z\right] = 0$

Problem. Use the angular momentum commutation relations to prove that $L^2 = L_x^2 + L_y^2 + L_z^2$ commutes with L_z.
Solution. Since L_z^2 clearly commutes with L_z,

$$\left[L^2, L_z\right] = \left[L_x^2, L_z\right] + \left[L_y^2, L_z\right].$$

Use $\left[A^2, B\right] = A\,[A, B] + [A, B]\,A$. Since $[L_x, L_z] = -\mathrm{i}\hbar L_y$,

$$\left[L_x^2, L_z\right] = -\mathrm{i}\hbar(L_x L_y + L_y L_x).$$

Since $[L_y, L_z] = \mathrm{i}\hbar L_x$,

$$\left[L_y^2, L_z\right] = \mathrm{i}\hbar(L_y L_x + L_x L_y).$$

The two terms cancel.

Final answer

$$\boxed{\left[L^2, L_z\right] = 0.}$$

Physical interpretation. A state can have definite total angular momentum magnitude and a definite z-component at the same time. This is why the labels l and m can be used simultaneously.

Example 3: Allowed m values for $l = 2$

Problem. For an orbital angular momentum state with $l = 2$, list all allowed m values and the corresponding L_z eigenvalues.
Solution. For fixed l, the magnetic quantum number takes integer steps from $-l$ to $+l$:

$$m = -l, -l+1, \ldots, l.$$

For $l = 2$,

$$m = -2, -1, 0, 1, 2.$$

The L_z eigenvalue is $m\hbar$, so the possible measured values are

$$-2\hbar,\ -\hbar,\ 0,\ \hbar,\ 2\hbar.$$

Final answer

$$\boxed{m = -2, -1, 0, 1, 2, \qquad L_z = m\hbar.}$$

Physical interpretation. A d-state has five possible orientations relative to a chosen axis. The word orientation must be understood quantum mechanically: only the projection is sharp.

Example 4: Magnitude of angular momentum for $l = 1$

Problem. Find the magnitude associated with orbital angular momentum when $l = 1$.
Solution. The eigenvalue of L^2 is

$$\hbar^2 l(l+1).$$

For $l = 1$,

$$L^2 = 2\hbar^2.$$

The vector-model magnitude is therefore

$$|\mathbf{L}| = \sqrt{2}\hbar.$$

Final answer

$$\boxed{|\mathbf{L}| = \sqrt{2}\,\hbar \quad \text{for } l = 1.}$$

Physical interpretation. The maximum projection is $+\hbar$, but the magnitude is $\sqrt{2}\hbar$. Even the highest m state is not a classical vector exactly aligned with the z-axis.

Example 5: Apply L_+ and L_- to $|l,m\rangle$

Problem. Compute $L_+ |2,1\rangle$ and $L_- |2,1\rangle$.
Solution. Use

$$L_\pm |l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}\, |l, m\pm 1\rangle .$$

For $L_+ |2,1\rangle$,

$$l(l+1) - m(m+1) = 2(3) - 1(2) = 4.$$

Thus

$$L_+ |2,1\rangle = 2\hbar\, |2,2\rangle .$$

For $L_- |2,1\rangle$,

$$l(l+1) - m(m-1) = 6 - 1(0) = 6,$$

so

$$L_- |2,1\rangle = \sqrt{6}\,\hbar\, |2,0\rangle .$$

Final answer

$$\boxed{L_+ |2,1\rangle = 2\hbar\, |2,2\rangle , \qquad L_- |2,1\rangle = \sqrt{6}\,\hbar\, |2,0\rangle .}$$

Physical interpretation. The raising operator increases m by one; the lowering operator decreases it by one. The square-root coefficient is fixed by normalization and by the angular momentum algebra.

Example 6: Construct L_x, L_y, L_z for $l = 1$

Problem. In the basis $(|1,1\rangle , |1,0\rangle , |1,-1\rangle)$, construct the matrices for L_x, L_y, L_z.
Solution. The diagonal matrix is

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

The ladder rules give

$$L_+ |1,0\rangle = \sqrt{2}\hbar\, |1,1\rangle , \qquad L_+ |1,-1\rangle = \sqrt{2}\hbar\, |1,0\rangle ,$$

with $L_+ |1,1\rangle = 0$. Therefore

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} , \qquad L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} .$$

Using $L_x = (L_+ + L_-)/2$ and $L_y = (L_+ - L_-)/(2\mathrm{i})$,

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \qquad L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} .$$

Final answer

$$\boxed{L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}.}$$

Physical interpretation. The basis diagonalizes L_z. The other components mix neighboring m values because they are built from raising and lowering operators.

Example 7: Eigenvalues and eigenvectors of σ_x

Problem. Find the eigenvalues and normalized eigenvectors of

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solution. Let the eigenvector be $\binom{a}{b}$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.$$

This gives $b = \lambda a$ and $a = \lambda b$, hence $\lambda^2 = 1$. For $\lambda = +1$, $b = a$, and normalization gives

$$|+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda = -1$, $b = -a$, and

$$|-x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Final answer

$$\boxed{\lambda = \pm 1, \qquad |\pm x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.}$$

Physical interpretation. A spin state with definite S_x is an equal-amplitude superposition of S_z up and down states.

Example 8: Measurement probabilities for a spin state

Problem. A spin-$\frac{1}{2}$ particle is prepared in

$$|\psi\rangle = \frac{\sqrt{3}}{2} |+\rangle + \frac{1}{2} |-\rangle.$$

What are the probabilities of measuring $S_z = +\hbar/2$ and $S_z = -\hbar/2$?

Solution. In the S_z basis, the amplitudes are

$$a = \frac{\sqrt{3}}{2}, \qquad b = \frac{1}{2}.$$

The probabilities are the squared magnitudes:

$$P(+\hbar/2) = |a|^2 = \frac{3}{4}, \qquad P(-\hbar/2) = |b|^2 = \frac{1}{4}.$$

They sum to one, as required.

Final answer

$$\boxed{P(S_z = +\hbar/2) = \frac{3}{4}, \qquad P(S_z = -\hbar/2) = \frac{1}{4}.}$$

Physical interpretation. The state is biased toward spin up along z, but it is not an eigenstate of S_z unless one coefficient has magnitude one and the other is zero.

Example 9: Add two spin-$\frac{1}{2}$ angular momenta

Problem. Construct the triplet and singlet states obtained by adding two spin-$\frac{1}{2}$ systems.
Solution. The allowed total angular momenta are

$$j = 1, \quad j = 0.$$

The highest state is unique:

$$|1,1\rangle = |+\rangle\,|+\rangle\,.$$

Lowering gives

$$J_-\,|1,1\rangle = \hbar\sqrt{2}\,|1,0\rangle\,.$$

But $J_- = S_{1-} + S_{2-}$, so

$$J_-\,|+\rangle\,|+\rangle = \hbar\,|-\rangle\,|+\rangle + \hbar\,|+\rangle\,|-\rangle\,.$$

Therefore

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|+\rangle\,|-\rangle + |-\rangle\,|+\rangle).$$

The lowest triplet state is

$$|1,-1\rangle = |-\rangle\,|-\rangle\,.$$

The remaining normalized $m = 0$ state orthogonal to $|1,0\rangle$ is

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|+\rangle\,|-\rangle - |-\rangle\,|+\rangle).$$

Final answer

$$\boxed{\begin{aligned} |1,1\rangle &= |+\rangle\,|+\rangle\,, \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle\,|-\rangle + |-\rangle\,|+\rangle), \\ |1,-1\rangle &= |-\rangle\,|-\rangle\,, \\ |0,0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle\,|-\rangle - |-\rangle\,|+\rangle). \end{aligned}}$$

Physical interpretation. The triplet is symmetric under exchange of the two spins. The singlet is antisymmetric and has zero total angular momentum.

Example 10: Clebsch–Gordan coefficients for two spin-$\frac{1}{2}$ particles

Problem. From the coupled states of two spin-$\frac{1}{2}$ particles, identify the nonzero Clebsch–Gordan coefficients for $m = 0$.

Solution. For $m = 0$, the allowed uncoupled product states are $|+\rangle|-\rangle$ and $|-\rangle|+\rangle$. The triplet and singlet states are

$$|1,0\rangle = \frac{1}{\sqrt{2}}|+\rangle|-\rangle + \frac{1}{\sqrt{2}}|-\rangle|+\rangle,$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}|+\rangle|-\rangle - \frac{1}{\sqrt{2}}|-\rangle|+\rangle.$$

Thus the coefficients are the amplitudes multiplying each uncoupled state.

Final answer

$$\boxed{\begin{aligned} C^{1\,0}_{\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,-\frac{1}{2}} &= \frac{1}{\sqrt{2}}, \quad C^{1\,0}_{\frac{1}{2}\,-\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}} = \frac{1}{\sqrt{2}}, \\ C^{0\,0}_{\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,-\frac{1}{2}} &= \frac{1}{\sqrt{2}}, \quad C^{0\,0}_{\frac{1}{2}\,-\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}} = -\frac{1}{\sqrt{2}}. \end{aligned}}$$

Physical interpretation. The relative sign distinguishes the symmetric triplet from the antisymmetric singlet. Both are built from the same two uncoupled $m = 0$ product states.

Example 11: Show that a central potential conserves angular momentum

Problem. For

$$H = \frac{p^2}{2m} + V(r),$$

show that angular momentum is conserved.

Solution. A central potential depends only on r, so it is unchanged under rotations. Since L_i generates rotations,

$$[V(r), L_i] = 0.$$

The kinetic energy $p^2/2m$ is also rotationally invariant, so

$$\left[p^2, L_i\right] = 0.$$

Therefore

$$[H, L_i] = 0$$

for $i = x, y, z$. In the Heisenberg picture,

$$\frac{\mathrm{d}L_i}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[H, L_i] = 0.$$

Final answer

$$\boxed{[H, \mathbf{L}] = 0, \quad \text{so } \mathbf{L} \text{ is conserved for a central potential.}}$$

Physical interpretation. Conservation of angular momentum is the quantum expression of rotational symmetry. No direction in space is preferred by $V(r)$.

Example 12: Hydrogen quantum numbers up to $n = 3$

Problem. List the allowed (n, l, m) values for hydrogen up to $n = 3$.
Solution. The rules are

$$n = 1, 2, \ldots, \qquad l = 0, 1, \ldots, n-1, \qquad m = -l, \ldots, l.$$

For $n = 1$, only $l = 0$, so $m = 0$:

$$(1, 0, 0).$$

For $n = 2$, $l = 0, 1$. Thus

$$(2, 0, 0), \qquad (2, 1, -1), (2, 1, 0), (2, 1, 1).$$

For $n = 3$, $l = 0, 1, 2$. Thus

$$(3, 0, 0),$$

$$(3, 1, -1), (3, 1, 0), (3, 1, 1),$$

$$(3, 2, -2), (3, 2, -1), (3, 2, 0), (3, 2, 1), (3, 2, 2).$$

Final answer

n	l	m
1	0	0
2	0	0
2	1	$-1, 0, 1$
3	0	0
3	1	$-1, 0, 1$
3	2	$-2, -1, 0, 1, 2$

Physical interpretation. Increasing n allows more orbital angular momentum values. Each l subspace contains $2l + 1$ magnetic substates.

Example 13: Degeneracy of the hydrogen level $n = 3$

Problem. Calculate the orbital degeneracy of the nonrelativistic hydrogen energy level with $n = 3$. Then include electron spin.
Solution. For a fixed n, allowed l values are $0, 1, \ldots, n-1$. For $n = 3$, these are $l = 0, 1, 2$. The number of m states for each l is $2l + 1$, so

$$g_{\text{orb}} = (2 \cdot 0 + 1) + (2 \cdot 1 + 1) + (2 \cdot 2 + 1) = 1 + 3 + 5 = 9.$$

Including spin-$\frac{1}{2}$ doubles the number of states:

$$g_{\text{spin+orb}} = 2 g_{\text{orb}} = 18.$$

Final answer

$$\boxed{g_{\text{orb}}(n = 3) = 9, \qquad g_{\text{including spin}}(n = 3) = 18.}$$

Physical interpretation. The $n = 3$ Coulomb energy level contains $3s$, $3p$, and $3d$ orbitals. Their m-sublevels are degenerate in the nonrelativistic Coulomb problem.

Example 14: Eigenfunctions of L_z

Problem. Show that the azimuthal function

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

is an eigenfunction of $L_z = -i\hbar\,\partial/\partial\phi$, and determine the allowed values of m from single-valuedness.

Solution. Acting with L_z,

$$L_z \Phi_m = -i\hbar \frac{\partial}{\partial\phi}\left(\frac{1}{\sqrt{2\pi}} e^{im\phi}\right) = -i\hbar(im)\Phi_m = m\hbar\Phi_m.$$

Therefore Φ_m is an eigenfunction with eigenvalue $m\hbar$. For an orbital wavefunction in ordinary space, the value at ϕ and $\phi + 2\pi$ must be the same:

$$\Phi_m(\phi + 2\pi) = \Phi_m(\phi).$$

This requires

$$e^{im(\phi+2\pi)} = e^{im\phi},$$

or

$$e^{2\pi i m} = 1.$$

Thus m must be an integer.

Final answer

$$\boxed{L_z\Phi_m = m\hbar\Phi_m, \qquad m = 0, \pm1, \pm2, \ldots.}$$

Physical interpretation. The quantization of L_z for orbital motion is tied to the single-valuedness of the spatial wavefunction around the azimuthal angle.

Example 15: Expectation values for a spinor

Problem. For

$$|\psi\rangle = \cos\frac{\theta}{2}\,|+\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|-\rangle\,,$$

calculate $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, and $\langle\sigma_z\rangle$.

Solution. In column form,

$$|\psi\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}.$$

Let $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$. Then

$$\langle\psi| = (c, e^{-i\phi}s).$$

For σ_x,

$$\langle\sigma_x\rangle = (c, e^{-i\phi}s)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c \\ e^{i\phi}s \end{pmatrix} = cs(e^{i\phi} + e^{-i\phi}) = 2cs\cos\phi.$$

Since $2cs = \sin\theta$,

$$\langle\sigma_x\rangle = \sin\theta\cos\phi.$$

Similarly,

$$\langle \sigma_y \rangle = (c, \mathrm{e}^{-\mathrm{i}\phi} s) \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} c \\ \mathrm{e}^{\mathrm{i}\phi} s \end{pmatrix} = \sin\theta \sin\phi.$$

For σ_z,

$$\langle \sigma_z \rangle = c^2 - s^2 = \cos\theta.$$

Final answer

$$\boxed{\langle \boldsymbol{\sigma} \rangle = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).}$$

Physical interpretation. The spinor maps to a point on the Bloch sphere. The vector gives expectation values of spin components, not simultaneous definite measurement results.

Example 16: Rotate a spin-up state about the y-axis

Problem. A spin-$\frac{1}{2}$ particle is initially in $|+\rangle$. Apply a rotation by angle θ about the y-axis and find the resulting spinor.

Solution. The spin-$\frac{1}{2}$ rotation operator about y is

$$U_y(\theta) = \exp\left(-\frac{\mathrm{i}\theta}{2}\sigma_y\right).$$

Since $\sigma_y^2 = I$,

$$U_y(\theta) = \cos\frac{\theta}{2} I - \mathrm{i}\sin\frac{\theta}{2}\sigma_y.$$

Using

$$\sigma_y |+\rangle = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{i} \end{pmatrix} = \mathrm{i}\,|-\rangle,$$

we get

$$U_y(\theta)|+\rangle = \cos\frac{\theta}{2}|+\rangle - \mathrm{i}\sin\frac{\theta}{2}(\mathrm{i}\,|-\rangle) = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle.$$

Final answer

$$\boxed{U_y(\theta)|+\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle.}$$

Physical interpretation. Spin-$\frac{1}{2}$ rotations use half-angles. A rotation through θ in physical space changes the spinor coefficients through $\theta/2$.

Example 17: Evaluate $\mathbf{L}\cdot\mathbf{S}$ using J^2

Problem. For an electron with $l = 1$ and $s = \frac{1}{2}$, find the eigenvalues of $\mathbf{L}\cdot\mathbf{S}$ for the two possible values of j.

Solution. Use

$$\mathbf{L}\cdot\mathbf{S} = \frac{1}{2}(J^2 - L^2 - S^2).$$

The possible total angular momenta are

$$j = l + s = \frac{3}{2}, \qquad j = l - s = \frac{1}{2}.$$

For $l = 1$,

$$L^2 = \hbar^2 l(l+1) = 2\hbar^2.$$

For $s = \frac{1}{2}$,

$$S^2 = \hbar^2 s(s+1) = \frac{3}{4}\hbar^2.$$

If $j = \frac{3}{2}$,

$$J^2 = \hbar^2 \frac{3}{2}\frac{5}{2} = \frac{15}{4}\hbar^2,$$

so

$$\mathbf{L}\cdot\mathbf{S} = \frac{1}{2}\left(\frac{15}{4} - 2 - \frac{3}{4}\right)\hbar^2 = \frac{1}{2}(1)\hbar^2 = \frac{1}{2}\hbar^2.$$

If $j = \frac{1}{2}$,

$$J^2 = \hbar^2 \frac{1}{2}\frac{3}{2} = \frac{3}{4}\hbar^2,$$

so

$$\mathbf{L}\cdot\mathbf{S} = \frac{1}{2}\left(\frac{3}{4} - 2 - \frac{3}{4}\right)\hbar^2 = -\hbar^2.$$

Final answer

$$\boxed{\mathbf{L}\cdot\mathbf{S} = \begin{cases} \frac{1}{2}\hbar^2, & j = \frac{3}{2}, \\ -\hbar^2, & j = \frac{1}{2}. \end{cases}}$$

Physical interpretation. Spin-orbit coupling separates the $p_{3/2}$ and $p_{1/2}$ levels because the relative alignment of $\mathbf{L}$ and $\mathbf{S}$ differs in the two total-j multiplets.

Example 18: Effective potential for a Coulomb field

Problem. Write the effective radial potential for a particle in

$$V(r) = -\frac{k}{r}$$

with orbital angular momentum l, and explain the behavior near $r = 0$.

Solution. The effective potential is

$$V_{\text{eff}}(r) = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}.$$

For the Coulomb potential,

$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{\hbar^2 l(l+1)}{2mr^2}.$$

If $l = 0$, there is no centrifugal barrier and the leading behavior near $r = 0$ is attractive like $-k/r$. If $l > 0$, the $1/r^2$ term dominates over the $1/r$ term at sufficiently small r, so

$$V_{\text{eff}}(r) \to +\infty \quad \text{as } r \to 0.$$

Final answer

$$\boxed{V_{\text{eff}}(r) = -\frac{k}{r} + \frac{\hbar^2 l(l+1)}{2mr^2}.}$$

For $l > 0$, the centrifugal barrier dominates near the origin.

Physical interpretation. Nonzero orbital angular momentum suppresses probability near the origin. This is why higher-l hydrogen orbitals penetrate the nucleus less strongly than s-orbitals.

Example 19: Possible j values for a d-electron

Problem. An electron has orbital angular momentum $l = 2$ and spin $s = \frac{1}{2}$. Find the possible total angular momentum quantum numbers j, the number of m_j states in each multiplet, and the total number of states.

Solution. The total angular momentum satisfies the triangle rule:

$$j = |l - s|,\ |l - s| + 1, \ldots, l + s.$$

With $l = 2$ and $s = \frac{1}{2}$,

$$j = \frac{3}{2}, \quad \frac{5}{2}.$$

For each j, the number of magnetic substates is $2j + 1$. Thus

$$j = \frac{5}{2} \quad \Rightarrow \quad 2j + 1 = 6,$$

and

$$j = \frac{3}{2} \quad \Rightarrow \quad 2j + 1 = 4.$$

The total number of coupled states is $6 + 4 = 10$. This agrees with the uncoupled counting:

$$(2l + 1)(2s + 1) = 5 \times 2 = 10.$$

Final answer

$$\boxed{j = \frac{5}{2}, \frac{3}{2}, \qquad \text{multiplet sizes 6 and 4,} \qquad \text{total states 10.}}$$

Physical interpretation. A d-electron has five orbital projection states and two spin projection states. Spin-orbit coupling reorganizes these ten states into total angular momentum multiplets of dimensions six and four.

Example 20: Spin-orbit eigenvalues for a d-electron

Problem. For the same $l = 2$, $s = \frac{1}{2}$ electron, find the eigenvalues of $\mathbf{L} \cdot \mathbf{S}$ for $j = \frac{5}{2}$ and $j = \frac{3}{2}$.

Solution. Use

$$\mathbf{L} \cdot \mathbf{S} = \frac{1}{2}(J^2 - L^2 - S^2).$$

For $l = 2$,

$$L^2 = \hbar^2 l(l + 1) = 6\hbar^2.$$

For $s = \frac{1}{2}$,

$$S^2 = \frac{3}{4}\hbar^2.$$

If $j = \frac{5}{2}$,

$$J^2 = \hbar^2 \frac{5}{2}\frac{7}{2} = \frac{35}{4}\hbar^2,$$

so

$$\mathbf{L}\cdot\mathbf{S}=\frac{1}{2}\left(\frac{35}{4}-6-\frac{3}{4}\right)\hbar^2=\hbar^2.$$

If $j=\frac{3}{2}$,

$$J^2=\hbar^2\frac{3}{2}\frac{5}{2}=\frac{15}{4}\hbar^2,$$

so

$$\mathbf{L}\cdot\mathbf{S}=\frac{1}{2}\left(\frac{15}{4}-6-\frac{3}{4}\right)\hbar^2=-\frac{3}{2}\hbar^2.$$

Final answer

$$\boxed{\mathbf{L}\cdot\mathbf{S}=\begin{cases}\hbar^2, & j=\frac{5}{2},\\ -\frac{3}{2}\hbar^2, & j=\frac{3}{2}.\end{cases}}$$

Physical interpretation. The two d-electron total angular momentum multiplets have different spin-orbit energies because the orbital and spin angular momenta have different relative coupling.

Example 21: Show that the singlet is annihilated by J

Problem. For two spin-$\frac{1}{2}$ particles, show that

$$|0,0\rangle=\frac{1}{\sqrt{2}}(|+\rangle|-\rangle-|-\rangle|+\rangle)$$

has total angular momentum zero.

Solution. First act with $J_z=S_{1z}+S_{2z}$. The state $|+\rangle|-\rangle$ has total $m=+\frac{1}{2}-\frac{1}{2}=0$, and $|-\rangle|+\rangle$ also has total $m=0$. Therefore

$$J_z|0,0\rangle=0.$$

Next act with $J_+=S_{1+}+S_{2+}$. Since $S_+|+\rangle=0$ and $S_+|-\rangle=\hbar|+\rangle$,

$$J_+|+\rangle|-\rangle=\hbar|+\rangle|+\rangle,$$

while

$$J_+|-\rangle|+\rangle=\hbar|+\rangle|+\rangle.$$

The two terms cancel in the antisymmetric combination, so

$$J_+|0,0\rangle=0.$$

Similarly, using $S_-|-\rangle=0$ and $S_-|+\rangle=\hbar|-\rangle$, one finds

$$J_-|0,0\rangle=0.$$

If J_z, J_+, and J_- annihilate the state, then all components of $\mathbf{J}$ annihilate it, and hence $J^2|0,0\rangle=0$.

Final answer

$$\boxed{J_z|0,0\rangle=J_+|0,0\rangle=J_-|0,0\rangle=0,\quad\text{so } J^2|0,0\rangle=0.}$$

Physical interpretation. The singlet is rotationally invariant. It has no preferred direction and no total angular momentum, even though each constituent particle has spin-$\frac{1}{2}$.

Example 22: A simple m-selection rule

Problem. Suppose an operator T_q satisfies

$$[L_z, T_q] = q\hbar T_q.$$

Show that $\langle l', m'| T_q |l, m\rangle$ can be nonzero only if $m' = m + q$.
Solution. Consider the matrix element of the commutator:

$$\langle l', m'| [L_z, T_q] |l, m\rangle .$$

Using the given commutator, this equals

$$q\hbar \langle l', m'| T_q |l, m\rangle .$$

Evaluating the commutator directly gives

$$\langle l', m'| L_z T_q |l, m\rangle - \langle l', m'| T_q L_z |l, m\rangle .$$

Since $L_z |l, m\rangle = m\hbar |l, m\rangle$ and $\langle l', m'| L_z = m'\hbar \langle l', m'|$, the expression is

$$(m' - m)\hbar \langle l', m'| T_q |l, m\rangle .$$

Therefore

$$(m' - m)\hbar \langle l', m'| T_q |l, m\rangle = q\hbar \langle l', m'| T_q |l, m\rangle .$$

For the matrix element to be nonzero, $m' - m = q$.

Final answer

$$\boxed{\langle l', m'| T_q |l, m\rangle \neq 0 \quad \Rightarrow \quad m' = m + q.}$$

Physical interpretation. The commutator with L_z tells how much magnetic quantum number the operator carries. This is the algebraic origin of many magnetic quantum number selection rules.

How to learn from these examples

The examples above are meant to train a small number of reusable habits. The first habit is to begin with commutators rather than with components written out in full, whenever possible. Direct coordinate calculations are sometimes necessary, as in the derivation of $[L_x, L_y]$, but most later results follow more cleanly from the algebra. For instance, once $[L_z, L_+] = \hbar L_+$ is known, the raising property follows immediately.

The second habit is to keep basis information explicit. A matrix is meaningful only after the basis and ordering are specified. The same abstract operator can have different-looking matrices in different bases. Similarly, a spinor column vector such as $(1, 0)^T$ means spin up only after the S_z basis has been chosen. Many mistakes in angular momentum problems are really basis mistakes.

The third habit is to distinguish sharp values from expectation values. In $|l, m\rangle$, L^2 and L_z are sharp. The components L_x and L_y have expectation value zero in many such states, but they are not sharply zero. In spin problems, a Bloch vector gives expectation values, while individual measurements still return discrete eigenvalues. Remembering this distinction prevents a slide back into an overly classical vector picture.

The fourth habit is to use symmetry before calculation. In a central potential, one knows immediately that m-states are degenerate. In a spin-orbit problem, one knows that $\mathbf{L} \cdot \mathbf{S}$ is diagonal in the coupled j-basis. In a transition problem, one can often identify forbidden transitions before doing radial integrals. Symmetry does not solve every detail, but it decides the structure of the calculation.

Common algebraic checks

Several checks are worth applying routinely. Ladder coefficients should vanish at the top and bottom of a multiplet. Matrix representations should be Hermitian for observables. The trace of J_z in a complete j-multiplet should be zero because the m-values are symmetric about zero. The dimension of coupled multiplets should equal the dimension of the uncoupled product space. Probabilities must sum to one. Hydrogen degeneracies should match $\sum_{l=0}^{n-1}(2l+1) = n^2$ before spin.

These checks are simple, but they catch sign errors and missing states. Angular momentum calculations often involve square roots and phases; a final answer may look plausible while still being wrong. Building consistency checks into the calculation is part of doing the subject well.

A final practice problem

As an additional exercise, consider a particle in a state with $l = 1$:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,|1,1\rangle + \frac{1}{\sqrt{2}}\,|1,-1\rangle\,.$$

It is normalized and has definite $L^2 = 2\hbar^2$. A measurement of L_z gives $+\hbar$ with probability $1/2$ and $-\hbar$ with probability $1/2$. The expectation value of L_z is zero, but the state is not an $m = 0$ state. This example is a compact reminder that expectation value zero and eigenvalue zero are very different statements.

Exam-style strategy

In timed problem solving, first identify the angular momentum type: orbital, spin, total, or a sum of two angular momenta. Then write down the relevant good quantum numbers. If the problem asks for possible values, use the spectrum rules. If it asks for an operator action, use ladder operators. If it asks for probabilities, identify the measurement basis. If it asks for energy shifts involving scalar products, rewrite the scalar product using total squared angular momentum.

This strategy prevents overcalculation. For instance, a problem asking for the possible j values from $l = 3$ and $s = \frac{1}{2}$ does not require Clebsch–Gordan tables; the triangle rule gives $j = 7/2, 5/2$. A problem asking for $\mathbf{L} \cdot \mathbf{S}$ eigenvalues does not require explicit wavefunctions; the identity involving J^2, L^2, S^2 is enough. A problem asking for L_z eigenvalues does not require solving the Schrodinger equation; the answer is $m\hbar$.

Common sources of wrong answers

The most common error is replacing $\sqrt{l(l+1)}\hbar$ by $l\hbar$ for the magnitude of angular momentum. The latter is the maximum projection, not the magnitude. Another common error is allowing m values outside the range $-l \le m \le l$. A third is forgetting that spin-$\frac{1}{2}$ has $s(s+1) = 3/4$, not $1/4$, in S^2. A fourth is using Clebsch–Gordan coefficients without checking the condition $m = m_1 + m_2$.

A useful defensive habit is to check dimensions and limits. Angular momentum eigenvalues should carry factors of $\hbar$ or $\hbar^2$. Probabilities should be dimensionless and sum to one. Degeneracies should be positive integers. Ladder coefficients should be real and nonnegative with the standard convention. These simple checks often reveal mistakes before they propagate through a longer solution.

1.14 Conceptual Summary

Angular momentum in quantum mechanics begins with a familiar classical expression but becomes a structural principle of the theory. Orbital angular momentum is represented by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

with $\mathbf{p} = -\mathrm{i}\hbar\nabla$ in the position representation. Its components are Hermitian operators, so they correspond to measurable quantities. Yet they do not behave like ordinary simultaneously knowable vector components, because

$$[L_i, L_j] = \mathrm{i}\hbar\epsilon_{ijk}L_k.$$

This single commutation relation is responsible for much of the chapter.

The first consequence is incompatibility. The components L_x, L_y, L_z do not commute with one another, so a quantum state cannot generally possess definite values of all three. However,

$$\left[L^2, L_i\right] = 0$$

for each component. Therefore one may choose simultaneous eigenstates of L^2 and one component, conventionally L_z. These states are labelled $|l, m\rangle$, with

$$L^2 \left|l, m\right\rangle = \hbar^2 l(l+1) \left|l, m\right\rangle, \qquad L_z \left|l, m\right\rangle = m\hbar \left|l, m\right\rangle.$$

For orbital angular momentum, $l = 0, 1, 2, \ldots$, and $m = -l, \ldots, l$. The number of m-states for fixed l is $2l + 1$.

The ladder operators $L_+ = L_x + \mathrm{i}L_y$ and $L_- = L_x - \mathrm{i}L_y$ explain the discrete structure. They raise or lower m by one unit while leaving l unchanged:

$$L_\pm \left|l, m\right\rangle = \hbar\sqrt{l(l+1) - m(m \pm 1)} \left|l, m \pm 1\right\rangle.$$

The ladder must terminate, and the termination points are $m = +l$ and $m = -l$. This algebraic argument gives the quantization of angular momentum without requiring explicit wavefunctions.

Spin is intrinsic angular momentum. It obeys the same commutation relations as orbital angular momentum, but it is not the literal rotation of a particle through ordinary space. A spin-$\frac{1}{2}$ system has a two-dimensional Hilbert space. Its spin operators are

$$S_i = \frac{\hbar}{2}\sigma_i,$$

where the Pauli matrices satisfy

$$[\sigma_i, \sigma_j] = 2\mathrm{i}\epsilon_{ijk}\sigma_k, \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}I.$$

Measurements of spin along any axis give only two outcomes, $+\hbar/2$ and $-\hbar/2$. The Stern–Gerlach experiment is the classic demonstration of this discreteness.

When a system contains more than one angular momentum, the total angular momentum is the vector sum:

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2.$$

The uncoupled basis $|j_1, m_1\rangle\,|j_2, m_2\rangle$ is useful for individual projections, while the coupled basis $|j, m\rangle$ is useful for total angular momentum. The allowed total quantum numbers are

$$j = |j_1 - j_2|, \ldots, j_1 + j_2, \qquad m = m_1 + m_2.$$

Clebsch–Gordan coefficients are the transformation amplitudes between these bases. They are essential whenever a Hamiltonian or interaction is simpler in terms of total angular momentum than in terms of individual components.

Angular momentum also generates rotations:

$$R(\hat{\mathbf{n}}, \theta) = \exp\left(-\frac{\mathrm{i}}{\hbar}\theta\, \hat{\mathbf{n}} \cdot \mathbf{J}\right).$$

This equation connects the algebra of angular momentum to the geometry of space and to the symmetry of physical laws. If a Hamiltonian is rotationally invariant, it commutes with the relevant angular momentum. Conservation of angular momentum is therefore the quantum expression of rotational symmetry.

Central potentials show the power of these ideas in wave mechanics. If $V = V(r)$, then H, L^2, and L_z commute, and the wavefunction separates as

$$\psi(r, \theta, \phi) = R(r) Y_l^m(\theta, \phi).$$

The spherical harmonics carry the angular momentum quantum numbers. The radial equation contains an effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2},$$

where the second term is the centrifugal barrier. Thus l affects both angular shape and radial behavior.

The hydrogen atom is the central example. Its Coulomb potential is rotationally symmetric, and its bound states are labelled by

$$n = 1, 2, \ldots, \qquad l = 0, \ldots, n-1, \qquad m = -l, \ldots, l.$$

The nonrelativistic energies depend only on n:

$$E_n = -\frac{13.6\,\text{eV}}{n^2}.$$

Angular momentum organizes the orbital labels s, p, d, f, explains magnetic degeneracies, and provides the language needed for spin-orbit coupling and fine structure.

The essential message is compact: angular momentum is quantized because its components form a noncommuting algebra; L^2 and one component can be known together; spin is an intrinsic representation of the same algebra; combined systems require coupled bases and Clebsch–Gordan coefficients; rotations are generated by angular momentum; and central-field quantum mechanics is naturally written in the language of angular momentum. Once these ideas are mastered, a large part of atomic, molecular, nuclear, and condensed-matter quantum mechanics becomes conceptually organized rather than a collection of separate facts.

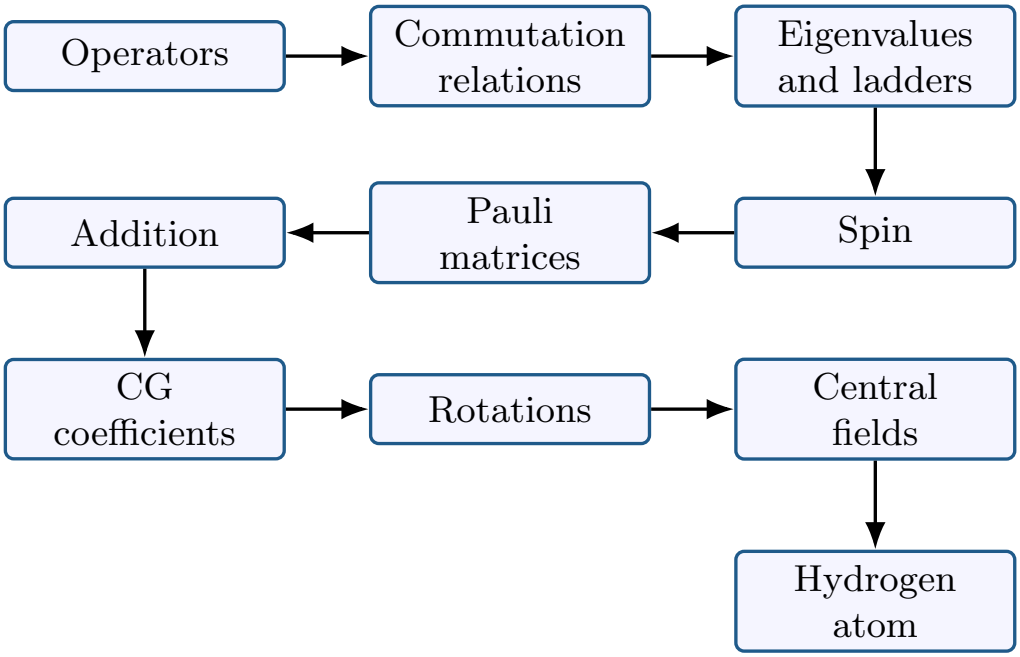

Figure 1.19: Conceptual flow of the chapter. Angular momentum begins as an operator algebra, becomes a theory of spectra and spin, and then organizes rotations, central potentials, and the hydrogen atom.

A final map of the ideas

The chapter can be read as a progression from algebra to physics. The algebra begins with commutators. Commutators determine which observables are compatible, which transformations are generated, and which quantities are conserved. From the angular momentum commutators come ladder operators, finite multiplets, and the eigenvalues $\hbar^2 j(j+1)$ and $m\hbar$. This is the abstract core.

The physical interpretations then branch in several directions. Orbital angular momentum describes angular structure of spatial wavefunctions. Spin describes internal angular momentum. Total angular momentum describes composite systems. Clebsch–Gordan coefficients translate between subsystem labels and total labels. Rotation operators show why angular momentum is tied to symmetry. Central potentials and hydrogen show how the same machinery organizes real wavefunctions and spectra.

When solving problems, the most important question is often not "Which formula should I use?" but "Which angular momentum is conserved?" If the Hamiltonian is spherically symmetric, use L^2 and L_z. If spin-orbit coupling matters, use J^2 and J_z. If individual spins are measured separately, use the uncoupled basis. If a magnetic field selects an axis, ask which component still commutes with the Hamiltonian. The right basis is the one chosen by symmetry and by the measurement being discussed.

The conceptual payoff is large. Angular momentum turns the complexity of three-dimensional quantum systems into a disciplined set of labels, rules, and transformations. It explains why spectra have patterns, why some transitions are absent, why spin measurements are discrete, and why the hydrogen atom is naturally described by n, l, m and then by j, m_j. The algebra is compact, but its consequences reach across nearly every part of quantum physics.

Checklist for mastery

A student who has mastered this chapter should be able to move comfortably among four representations of the same idea. In vector-operator language, they should know the commutators of J_x, J_y, J_z, the definition of J^2, and the meaning of compatible observables. In ladder-operator language, they should be able to derive allowed m-values, apply $J_\pm$, and explain why the ladder terminates. In matrix language, they should be able to construct spin-$\frac{1}{2}$ and spin-one matrices and use them to compute measurement probabilities. In wavefunction language, they should recognize spherical harmonics as orbital angular momentum eigenfunctions.

They should also be able to explain the physics in words. Angular momentum is quantized, but not because a classical arrow is forced to point only in certain directions. Components fail to commute, so only J^2 and one component can be sharp together. Spin is real angular momentum, but it is intrinsic rather than a literal rotation of an extended body. Clebsch–Gordan coefficients are not mysterious constants; they are change-of-basis amplitudes between product states and states of good total angular momentum.

Finally, they should know how to diagnose symmetry. If the Hamiltonian is rotationally invariant, angular momentum is conserved. If a field or perturbation selects an axis, some degeneracy may be lifted. If the potential is central, the angular part separates and Y_l^m appears. If the system is hydrogenic, the Coulomb potential adds special degeneracies and the quantum numbers n, l, m organize the spectrum. These are the habits that turn angular momentum from a collection of formulas into a working method.

Where to go next

The natural next step is tensor operators and the Wigner–Eckart theorem. That framework turns the selection-rule ideas in this chapter into a systematic method for evaluating angular parts of matrix elements. After that, one can study fine structure, hyperfine structure, and transition rates with much less brute-force integration. Another direction is representation theory, where angular momentum multiplets are treated as representations of SU(2) and SO(3). This gives a deeper mathematical explanation of why the allowed j values are integer or half-integer.

From the physical side, central potentials lead naturally to scattering theory and partial waves. Spin leads naturally to magnetic resonance, quantum information, and relativistic wave equations. Hydrogen leads naturally to many-electron atoms and spectroscopy. These topics may look different, but the same angular momentum machinery keeps reappearing. That recurrence is the best evidence that angular momentum is not a special trick for one chapter, but one of the central organizing languages of quantum mechanics.