B
Kinetic Energy in Generalized Coordinates

For a system of N point particles in three spatial dimensions the kinetic energy expressed in Cartesian coordinates is always given by

$$T = T(\dot{\boldsymbol{r}}) = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i^2 = \frac{1}{2}\sum_{i=1}^{3N} m_i \dot{x}_i^2 \tag{B.1}$$

with x_i ($i = 1, \ldots, 3N$) being the ith Cartesian component. For the masses m_i an obvious notation has been employed. If there are R constraints on the system, the Cartesian coordinates may be replaced by $f = 3N - R$ suitably chosen generalized coordinates q_k according to Eq. (2.44), or in a more compact notation

$$x_i = x_i(q, t), \qquad \forall i = 1, 2, \ldots, 3N \tag{B.2}$$

This coordinate transformation does *not* in general depend on the velocities $\dot{q}$. Accordingly the total time derivatives of the Cartesian coordinates read

$$\dot{x}_i = \frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{k=1}^{f} \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t} = \dot{x}_i(q, \dot{q}, t), \qquad \forall i = 1, 2, \ldots, 3N \tag{B.3}$$

and depend on both the generalized coordinates and velocities and the time. Inserting this expression into (B.1) one obtains

$$T = \frac{1}{2}\sum_{k,l=1}^{f} \underbrace{\sum_{i=1}^{3N} m_i \left(\frac{\partial x_i}{\partial q_k}\right)\left(\frac{\partial x_i}{\partial q_l}\right)}_{m_{kl}(q,t)} \dot{q}_k \dot{q}_l + \sum_{k=1}^{f} \underbrace{\sum_{i=1}^{3N} m_i \left(\frac{\partial x_i}{\partial q_k}\right)\left(\frac{\partial x_i}{\partial t}\right)}_{a_k(q,t)} \dot{q}_k$$

$$+ \frac{1}{2}\underbrace{\sum_{i=1}^{3N} m_i \left(\frac{\partial x_i}{\partial t}\right)^2}_{b(q,t)} = T(q, \dot{q}, t) \tag{B.4}$$

This is the most general expression for the kinetic energy T in terms of the generalized coordinates. As soon as other than Cartesian coordinates are em-

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ployed, T may no longer depend only on the velocities but also on the coordinates itself. Because of its definition by Eq. (B.4) the *mass matrix* is symmetric, $m_{kl} = m_{lk}$.

Equation (B.4) is simplified to a large extent if the constraints do not explicitly depend on time, i.e., the coordinate transformation given by Eq. (B.2) itself does not explicitly depend on time,

$$x_i \;=\; x_i(q(t)), \qquad \forall i = 1, 2, \ldots, 3N \tag{B.5}$$

Then the partial derivatives $\partial x_i / \partial t$ vanish and the kinetic energy is given by

$$T(q, \dot{q}) \;=\; \frac{1}{2} \sum_{k,l=1}^{f} m_{kl}(q)\, \dot{q}_k \dot{q}_l \tag{B.6}$$

If furthermore the potential U does *not* depend on the velocities, $U = U(q, t)$, one finds

$$\sum_{k=1}^{f} \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k \;=\; \sum_{k=1}^{f} \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k \;\stackrel{(B.6)}{=}\; \sum_{k,l=1}^{f} m_{kl} \dot{q}_k \dot{q}_l \;=\; 2T(q, \dot{q}) \tag{B.7}$$

Under those circumstances the energy of the system is thus given by

$$\sum_{k=1}^{f} \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L \;\stackrel{(B.7)}{=}\; 2T - T + U \;=\; T + U \;=\; E \tag{B.8}$$