C
Technical Proofs for Special Relativity

C.1
Invariance of Space-Time Interval

In section 3.1.2 we found the invariance under Lorentz transformations of the squared space-time interval s_{12}^2 between two events connected by a light signal being solely based on the relativity principle of Einstein, i.e., the constant speed of light in all inertial frames, cf. Eq. (3.5),

$$s_{12}^2 = s_{12}'^2 = 0 , \qquad \forall \text{ IS}' \qquad \text{(for light signal)} \tag{C.1}$$

As a consequence of Eq. (C.1) and the homogeneity of space and time and the isotropy of space, we now formally prove the invariance of the space-time interval for *any* two events E_1 and E_2, cf. Eq. (3.6),

$$s_{12}^2 = s_{12}'^2 , \qquad \forall \text{ IS}' \qquad \text{(for any two events)} \tag{C.2}$$

Proof: We thus consider these two arbitrary events with reference to two inertial frames IS and IS′ moving with velocity $\boldsymbol{v}_1$ relative to each other. We can always express the relationship between the four-dimensional distances s_{12}^2 and $s_{12}'^2$ between these two events as

$$s_{12}'^2 = F_{\boldsymbol{v}_1}\left(s_{12}^2\right) \tag{C.3}$$

where $F_{\boldsymbol{v}_1}(s)$ shall be a sufficiently smooth function permitting its Taylor expansion around $s = 0$, i.e.,

$$s_{12}'^2 \approx A(\boldsymbol{v}_1) + B(\boldsymbol{v}_1)\, s_{12}^2 + \frac{1}{2} C(\boldsymbol{v}_1) \left(s_{12}^2\right)^2 + \ldots \tag{C.4}$$

Eq. (C.4) must hold for *any* two events and thus also for events connected by a light signal. From Eq. (C.1) we thus find $A(\boldsymbol{v}_1) = 0$. Furthermore, Eq. (C.4) must also hold for events in the infinitesimal neighborhood of each other where $s_{12}^2 = \mathrm{d}s^2$ is an infinitesimal, i.e., very small quantity. For those events we thus arrive at

$$\mathrm{d}s'^2 = B(\boldsymbol{v}_1)\, \mathrm{d}s^2 \tag{C.5}$$

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Because of the assumption of a homogeneous space-time the function $B(\boldsymbol{v}_1)$ cannot depend on the space-time coordinates t and $\boldsymbol{r}$, and because of the assumption of spatial rotational invariance (isotropy of space) the function $B(\boldsymbol{v}_1)$ must *not* depend on the direction of $\boldsymbol{v}_1$ but only on its magnitude $v_1 = |\boldsymbol{v}_1|$, i.e.,

$$ds'^2 = B(v_1)\, ds^2 \tag{C.6}$$

The last ingredient for this proof is the group structure of Lorentz transformations, i.e., the possibility to consider a further inertial frame IS$''$ which moves with velocity $\boldsymbol{v}_2$ relative to IS. Its relative velocity against IS$'$ shall be $\boldsymbol{v}_{12}$. We thus arrive at the following relations for the squared space-time intervals between our two infinitesimally neighboring events,

$$ds'^2 = B(v_1)\, ds^2 \tag{C.7}$$

$$ds''^2 = B(v_2)\, ds^2 \tag{C.8}$$

$$ds''^2 = B(v_{12})\, ds'^2 \tag{C.9}$$

A simple rearrangement of Eqs. (C.7)–(C.9) yields

$$B(v_{12}) = \frac{B(v_2)}{B(v_1)} \tag{C.10}$$

The term on the left hand side of this equation is a function of the *angle* between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, cf. section 3.2.3, whereas the term on the right hand side is not. Eq. (C.10) can thus only be valid in general if $B(v_{12})$ is also *not* a function of the angle between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Since we have made no further assumptions for both the size and the orientation of the velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ this immediately implies

$$B(v) = const. = 1\,, \qquad \forall\, \text{velocities } v \tag{C.11}$$

This finally yields the desired result for infinitesimally neighboring events,

$$ds'^2 = ds^2 \tag{C.12}$$

and since Eq. (C.12) holds for *any* infinitesimally neighboring events within the whole space-time, it holds for any two events and thus Eq. (C.2) is proven.

C.2
Uniqueness of Lorentz Transformations

In section 3.1.3 we mentioned that Lorentz transformations of the form as given by Eq. (3.12) are the only *nonsingular* coordinate transformations from

IS to IS′, i.e., $x \rightarrow x'(x)$, that leave the four-dimensional space-time interval $\mathrm{d}s^2$ invariant. Nonsingular in this context means that both $x' = x'(x)$ and $x = x(x')$ are sufficiently smooth and well-behaved functions that feature a well-defined inverse.

Proof: We consider an arbitrary nonsingular coordinate transformation $x \rightarrow x'(x)$ and calculate the four-dimensional distance $\mathrm{d}s'^2$ between two infinitesimally neighboring events,

$$\begin{aligned} \mathrm{d}s'^2 &= g_{\alpha\beta}\,\mathrm{d}x'^\alpha\,\mathrm{d}x'^\beta = g_{\alpha\beta}\,\frac{\partial x'^\alpha}{\partial x^\mu}\,\frac{\partial x'^\beta}{\partial x^\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\ &= g_{\alpha\beta}\,(\partial_\mu x'^\alpha)\,(\partial_\nu x'^\beta)\,\mathrm{d}x^\mu \mathrm{d}x^\nu \end{aligned} \tag{C.13}$$

where we have taken advantage of the relation $\mathrm{d}x'^\alpha = (\partial_\mu x'^\alpha)\mathrm{d}x^\mu$ for infinitesimal space-time distances (at the second equality). Since we are looking for transformations that leave the space-time interval invariant for *any* two events,

$$\mathrm{d}s'^2 \overset{!}{=} \mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{C.14}$$

this yields the condition

$$g_{\mu\nu} = g_{\alpha\beta}\,(\partial_\mu x'^\alpha)\,(\partial_\nu x'^\beta) \tag{C.15}$$

We now differentiate this equation with reference to the arbitrary space-time component x^σ and, by the product rule, arrive at

$$0 = g_{\alpha\beta}\,(\partial_\mu \partial_\sigma x'^\alpha)\,(\partial_\nu x'^\beta) + g_{\alpha\beta}\,(\partial_\mu x'^\alpha)\,(\partial_\nu \partial_\sigma x'^\beta) \tag{C.16}$$

In order to solve for the second derivatives we write down Eq. (C.16) with indices μ and σ interchanged,

$$0 = g_{\alpha\beta}\,(\partial_\mu \partial_\sigma x'^\alpha)\,(\partial_\nu x'^\beta) + g_{\alpha\beta}\,(\partial_\sigma x'^\alpha)\,(\partial_\nu \partial_\mu x'^\beta) \tag{C.17}$$

and again with indices ν and σ interchanged,

$$0 = g_{\alpha\beta}\,(\partial_\mu \partial_\nu x'^\alpha)\,(\partial_\sigma x'^\beta) + g_{\alpha\beta}\,(\partial_\mu x'^\alpha)\,(\partial_\nu \partial_\sigma x'^\beta) \tag{C.18}$$

We now calculate (C.16) + (C.17) − (C.18) and, bearing in mind that α and β are just dummy indices which are summed over, we find

$$0 = 2\,g_{\alpha\beta}\,(\partial_\mu \partial_\sigma x'^\alpha)\,(\partial_\nu x'^\beta) \tag{C.19}$$

Since we have assumed a nonsingular coordinate transformation, the last term $(\partial_\nu x'^\beta)$ *cannot* vanish identically, and thus Eq. (C.19) can only hold in general if the second derivatives identically vanish, i.e.,

$$\partial_\mu \partial_\sigma x'^\alpha = 0 \tag{C.20}$$

This immediately implies a linear coordinate transformation of the form as given by Eq. (3.12), i.e.,

$$x' \;=\; \Lambda x \,+\, a \qquad \Longleftrightarrow \qquad x'^{\mu} \;=\; \Lambda^{\mu}{}_{\nu}\, x^{\nu} \,+\, a^{\mu} \tag{C.21}$$

where the entries of the matrix Λ and the 4-vector a must be constants which do *not* depend on x. Insertion of Eq. (C.21) in Eq. (C.15) yields,

$$g_{\mu\nu} \;=\; g_{\alpha\beta}\,(\partial_{\mu} x'^{\alpha})\,(\partial_{\nu} x'^{\beta}) \;=\; g_{\alpha\beta}\,\Lambda^{\alpha}{}_{\mu}\,\Lambda^{\beta}{}_{\nu} \tag{C.22}$$

i.e., the fundamental property of Lorentz transformations as given by Eq. (3.17) is recoverd. This completes the proof of the uniqueness of Lorentz transformations as the natural symmetry transformations within the four-dimensional Minkowski space equipped with the metric g.

C.3
Useful Trigonometric and Hyperbolic Formulae for Lorentz Transformations

The trigonometric functions may be related to the exponential function via Euler's relation

$$e^{\mathrm{i}x} \;=\; \cos x \,+\, \mathrm{i}\sin x\,, \quad \forall x \in \mathbb{R} \tag{C.23}$$

which may be inverted to yield

$$\cos x \;=\; \tfrac{1}{2}\left(e^{\mathrm{i}x} + e^{-\mathrm{i}x}\right), \quad \sin x \;=\; \tfrac{1}{2\mathrm{i}}\left(e^{\mathrm{i}x} - e^{-\mathrm{i}x}\right), \quad \forall x \in \mathbb{R} \tag{C.24}$$

The tangent tan and cotangent cot are defined as

$$\tan x \;=\; \frac{\sin x}{\cos x}\,, \quad \forall x \in \mathbb{R} \setminus \left\{(2k+1)\tfrac{\pi}{2}\,,\; k \in \mathbb{Z}\right\} \tag{C.25}$$

$$\cot x \;=\; \frac{\cos x}{\sin x} \;=\; \frac{1}{\tan x}\,, \quad \forall x \in \mathbb{R} \setminus \left\{k\pi\,,\; k \in \mathbb{Z}\right\} \tag{C.26}$$

The inverse functions of the trigonometric functions (on suitably chosen domains) are denoted as $\arccos x$, $\arcsin x$, $\arctan x$, and $\operatorname{arccot} x$. Important relations between the trigonometric functions and/or their inverse functions are

$$\cos^2 x \,+\, \sin^2 x \;=\; 1\,, \quad \forall x \in \mathbb{R} \tag{C.27}$$

$$\arcsin x \;=\; \arctan \frac{x}{\sqrt{1-x^2}}\,, \quad \forall x \in]-1,1[\tag{C.28}$$

$$\arccos x \;=\; \operatorname{arccot} \frac{x}{\sqrt{1-x^2}}\,, \quad \forall x \in]-1,1[\tag{C.29}$$

$$\arctan x + \arctan y \;=\; \arctan\left(\frac{x+y}{1-xy}\right), \quad \forall x, y \in \mathbb{R} \text{ with } xy < 1 \tag{C.30}$$

Similarly to the definition of the usual trigonometric functions, the hyperbolic functions are given by

$$\cosh x = \tfrac{1}{2}\left(e^{x} + e^{-x}\right), \quad \sinh x = \tfrac{1}{2}\left(e^{x} - e^{-x}\right), \quad \forall x \in \mathbb{R} \tag{C.31}$$

and

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}, \quad \forall x \in \mathbb{R} \tag{C.32}$$

As a direct consequence it holds that

$$\cosh^2 x - \sinh^2 x = 1, \quad \forall x \in \mathbb{R} \tag{C.33}$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y, \quad \forall x \in \mathbb{R} \tag{C.34}$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y, \quad \forall x \in \mathbb{R} \tag{C.35}$$

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}, \quad \forall x \in \mathbb{R} \tag{C.36}$$

The inverse hyperbolic functions (on suitably chosen domains) are the *area-functions* arcosh, arsinh, and artanh. They satisfy

$$\operatorname{artanh} x = \operatorname{arcosh}\left(\frac{1}{\sqrt{1-x^2}}\right), \quad \forall x \in]-1,1[\tag{C.37}$$

$$\operatorname{artanh} x \pm \operatorname{artanh} y = \operatorname{artanh}\left(\frac{x \pm y}{1 \pm xy}\right), \quad \forall x, y \in \mathbb{R} \text{ with } xy < 1 \tag{C.38}$$