DERIVATION OF LAGRANGE'S EQUATIONS

Consider a system of particles: $k = 1, \ldots, N_P$. Let $\mathbf{r}^{(k)}$ denote the position of the k^{th} particle relative to an inertial reference frame. Let $\mathbf{f}^{(k)}$ be the resultant force acting on the mass m_k of the k^{th} particle. Then, from Newton's Second Law:

$$\mathbf{f}^{(k)} = m_k \ddot{\mathbf{r}}^{(k)} \quad k = 1, \ldots, N_P \tag{1}$$

or,

$$\mathbf{f}^{(k)} - m_k \ddot{\mathbf{r}}^{(k)} = \mathbf{0} \quad k = 1, \ldots, N_P \tag{1$'$}$$

Let $\delta\mathbf{r}^{(k)}$ be a 'virtual' displacement that is compatible with all constraints on the motion of the k^{th} particle. Clearly, any motion $\mathbf{r}^{(k)}$ that satisfies $(1)'$ satisfies

$$\sum_{k=1}^{N_P} (\mathbf{f}^{(k)} - m_k \ddot{\mathbf{r}}^{(k)}) \cdot \delta\mathbf{r}^{(k)} = 0 \tag{1$''$}$$

for all $\delta\mathbf{r}^{(k)}$. Equation $(1)''$ is, for dynamical systems, the analog of the statement that the virtual work of a system in equilibrium vanishes for all compatible virtual displacements of the system.

Consider a decomposition of the resultant force $\mathbf{f}^{(k)}$ of the form

$$\mathbf{f}^{(k)} = \mathbf{F}^{(k)} + \mathbf{R}^{(k)} \quad k = 1, \ldots, N_P$$

where $\mathbf{R}^{(k)}$ is the force due to the constraints and $\mathbf{F}^{(k)}$ is the resultant of all other forces acting on the k^{th} particle. Then, because the constraint forces $\mathbf{R}^{(k)}$ do no work during motions compatible with the constraints, i.e. $\mathbf{R}^{(k)} \cdot \delta\mathbf{r}^{(k)} = 0$, Eqn. $(1)''$ becomes

$$\sum_{k=1}^{N_P} (\mathbf{F}^{(k)} - m_k \ddot{\mathbf{r}}^{(k)}) \cdot \delta\mathbf{r}^{(k)} = 0 \tag{1$'''$}$$

for all compatible $\delta\mathbf{r}^{(k)}$. Equation $(1)'''$ is the form of Newton's Second Law for constrained dynamical systems that we will use in deriving Lagrange's equations.

For holonomic dynamical systems with n degrees of freedom all displacements $\mathbf{r}^{(k)}$ can be expressed in terms of n independent generalized coordinates $q_1(t), \ldots, q_n(t)$ by means of transformation equations of the form

$$\mathbf{r}^{(k)} = \mathbf{r}^{(k)}(q_1(t), \ldots, q_n(t), t) \quad k = 1, \ldots, N_P$$

Consequently, compatible virtual displacements $\delta \mathbf{r}^{(k)}$ can be expressed in terms of virtual changes δq_j in generalized coordinates (with time held constant) by

$$\delta \mathbf{r}^{(k)} = \sum_{j=1}^{n} \frac{\partial \mathbf{r}^{(k)}}{\partial q_j} \delta q_j .$$

Then, the virtual work δW associated with the virtual displacements $\delta \mathbf{r}^{(k)}$ can be written as

$$\delta W \equiv \sum_{k=1}^{N_P} \mathbf{F}^{(k)} \cdot \delta \mathbf{r}^{(k)} = \sum_{k=1}^{N_P} \sum_{j=1}^{n} \mathbf{F}^{(k)} \cdot \frac{\partial \mathbf{r}^{(k)}}{\partial q_j} \delta q_j$$

or

$$\delta W = \sum_{j=1}^{n} Q_j \delta q_j$$

where

$$Q_j \equiv \sum_{k=1}^{N_P} \mathbf{F}^{(k)} \cdot \frac{\partial \mathbf{r}^{(k)}}{\partial q_j}$$

are the generalized forces corresponding to the generalized coordinates q_j. Expressing the virtual work term in Eqn. $(1)'''$ in terms of generalized forces and displacements we can rewrite $(1)'''$ in the form

$$\sum_{k=1}^{N_P} m_k \ddot{\mathbf{r}}^{(k)} \cdot \delta \mathbf{r}^{(k)} = \sum_{j=1}^{n} Q_j \delta q_j$$

or,

$$\sum_{k=1}^{N_P} m_k \ddot{\mathbf{r}}^{(k)} \cdot \sum_{j=1}^{n} \frac{\partial \mathbf{r}^{(k)}}{\partial q_j} \delta q_j = \sum_{j=1}^{n} Q_j \delta q_j$$

or, interchanging the order of summation,

$$\sum_{j=1}^{n} [\sum_{k=1}^{N_P} m_k \ddot{\mathbf{r}}^{(k)} \cdot \frac{\partial \mathbf{r}^{(k)}}{\partial q_j}] \delta q_j = \sum_{j=1}^{n} Q_j \delta q_j . \tag{2}$$

To derive Lagrange's equations we want to show that the term in [ ] can be expressed in terms of derivatives of the kinetic energy

$$T = \frac{1}{2} \sum_{k=1}^{N_P} m_k \dot{\mathbf{r}}^{(k)} \cdot \dot{\mathbf{r}}^{(k)} . \tag{3}$$

To this end, differentiate Eqn. (3) with respect to $\dot{q}_j$:

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{k=1}^{N_P} m_k \dot{\mathbf{r}}^{(k)} \cdot \frac{\partial \dot{\mathbf{r}}^{(k)}}{\partial \dot{q}_j}. \tag{4}$$

Perhaps the least intuitive step in the derivation of Lagrange's equations is noting that the last term in Eqn. (4) satisfies the following identity, known as the "cancellation of dots".

$$\frac{\partial \dot{\mathbf{r}}^{(k)}}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}^{(k)}}{\partial q_j}. \tag{5}$$

We digress briefly to derive this equation. Differentiation of the transformation equations gives

$$\dot{\mathbf{r}}^{(k)} = \sum_{j=1}^{n} \frac{\partial \mathbf{r}^{(k)}}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}^{(k)}}{\partial t} \tag{6}$$

which shows that the function $\dot{\mathbf{r}}^{(k)}$ can be expressed as a function of the following form

$$\dot{\mathbf{r}}^{(k)} = \dot{\mathbf{r}}^{(k)}(q_1(t), \ldots, q_n(t), \dot{q}_1(t), \ldots, \dot{q}_n(t), t). \tag{7}$$

Differentiation of (7) with respect to time gives

$$\frac{d\dot{\mathbf{r}}^{(k)}}{dt} = \sum_{j=1}^{n} \frac{\partial \dot{\mathbf{r}}^{(k)}}{\partial q_j} \dot{q}_j + \frac{\partial \dot{\mathbf{r}}^{(k)}}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial \dot{\mathbf{r}}^{(k)}}{\partial t} \tag{8}$$

whereas differentiation of (6) with respect to time gives

$$\frac{d\dot{\mathbf{r}}^{(k)}}{dt} = \sum_{j=1}^{n} \frac{\partial \dot{\mathbf{r}}^{(k)}}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}^{(k)}}{\partial q_j} \ddot{q}_j + \frac{\partial \dot{\mathbf{r}}^{(k)}}{\partial t} \tag{9}$$

The first and third terms in Eqns. (8) and (9) are identical. The remaining terms can be the same for all motions only if Eqn. (5) holds. Thus, the "cancellation of dots" identity is proved.

Returning to the derivation of Lagrange's equations and using the "cancellation of dots" identity in Eqn.(4) we find

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{k=1}^{N_P} m_k \dot{\mathbf{r}}^{(k)} \cdot \frac{\partial \mathbf{r}^{(k)}}{\partial q_j}. \tag{10}$$

Differentiation of Eqn.(10) with respect to time gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \sum_{k=1}^{N_P} m_k \ddot{\mathbf{r}}^{(k)} \cdot \frac{\partial \mathbf{r}^{(k)}}{\partial q_j} + \sum_{k=1}^{N_P} m_k \dot{\mathbf{r}}^{(k)} \cdot \frac{\partial \dot{\mathbf{r}}^{(k)}}{\partial q_j}. \tag{11}$$

where the last term is simply $\partial T/\partial q_j$. Hence,

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = \sum_{k=1}^{N_P} m_k \ddot{\mathbf{r}}^{(k)} \cdot \frac{\partial \mathbf{r}^{(k)}}{\partial q_j}. \tag{12}$$

Substitution of (12) into Eqn.(2) gives

$$\sum_{j=1}^{n} [\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} - Q_j] \delta q_j = 0. \tag{13}$$

Because the variations δq_j in Eqn. (13) are independent, we conclude that

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} - Q_j = 0 \quad j = 1, \ldots, n. \tag{14}$$

Equations (14) are Lagrange's equations for systems with holonomic constraints, whether or not the forces are conservative.

If all forces are conservative, then $Q_j = -\partial V/\partial q_j$ and (14) becomes

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 \quad j = 1, \ldots, n$$

or,

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial (T - V)}{\partial q_j} = 0 \quad j = 1, \ldots, n$$

or, because $\partial V/\partial \dot{q}_j = 0$,

$$\frac{d}{dt}(\frac{\partial (T - V)}{\partial \dot{q}_j}) - \frac{\partial (T - V)}{\partial q_j} = 0 \quad j = 1, \ldots, n. \tag{15}$$

Finally, introducing the Lagrangian $L \equiv T - V$, Eqns. (15) can be written as

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = 0 \quad j = 1, \ldots, n. \tag{16}$$

Equations (16) are Lagrange's equations for conservative systems with holonomic constraints.