# D
# Relations for Pauli and Dirac Matrices

## D.1
## Pauli Spin Matrices

The Pauli spin matrices introduced in Eq. (4.140) fulfill some important relations. First of all, the squared matrices yield the (2×2) unit matrix $\mathbf{1}_2$,

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}_2 \tag{D.1}$$

which is an essential property when calculating the square of the spin operator. Next, multiplication of two different Pauli spin matrices yields the third one multiplied by the (positive or negative) imaginary unit,

$$\sigma_x\sigma_y = \mathrm{i}\sigma_z \quad , \quad \sigma_x\sigma_z = -\mathrm{i}\sigma_y \quad , \quad \sigma_y\sigma_z = \mathrm{i}\sigma_x \tag{D.2}$$

$$\sigma_y\sigma_x = -\mathrm{i}\sigma_z \quad , \quad \sigma_z\sigma_x = \mathrm{i}\sigma_y \quad , \quad \sigma_z\sigma_y = -\mathrm{i}\sigma_x \tag{D.3}$$

This may be expressed in more compact form for all cyclic permutations of $i, j, k \in \{1, 2, 3\}$ as

$$\sigma_i\sigma_j \; = \; \delta_{ij}\mathbf{1}_2 \; + \; \mathrm{i}\sum_{k=1}^{3} \varepsilon_{ijk}\sigma_k \tag{D.4}$$

where $\{1, 2, 3\}$ and $\{x, y, z\}$ are used synonymously. As a direct consequence of Eq. (D.4) the commutation and anticommutation relations for Pauli spin matrices are given by

$$\left[\sigma_i\, ,\, \sigma_j\right] \; = \; 2\mathrm{i}\sum_{k=1}^{3} \varepsilon_{ijk}\sigma_k \qquad \text{and} \qquad \left\{\sigma_i\, ,\, \sigma_j\right\} \; = \; 2\delta_{ij}\mathbf{1}_2 \tag{D.5}$$

These relations may be generalized to the four-component case if we consider the even matrix $\mathbf{\Sigma}$ and the Dirac matrices $\boldsymbol{\alpha}$ and $\beta$; cf. chapter 5, for which we have

$$\alpha_x^2 \; = \; \alpha_y^2 \; = \; \alpha_z^2 \; = \; \beta^2 \; = \; \mathbf{1}_4 \tag{D.6}$$

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$$\alpha_i\alpha_j = \mathbf{1}_2 \otimes \sigma_i\sigma_j = \begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{pmatrix} \tag{D.7}$$

so that commutators and anticommutators read

$$\left[\alpha_i\,,\alpha_j\right] = 2\mathrm{i}\sum_{k=1}^{3}\varepsilon_{ijk}\Sigma_k \tag{D.8}$$

$$\left\{\alpha_i\,,\alpha_j\right\} = 2\delta_{ij}\mathbf{1}_4 \qquad \text{and} \qquad \left\{\alpha_i\,,\beta\right\} = 0 \tag{D.9}$$

The tensor product denoted by '$\otimes$' is to be evaluated according to the general prescription

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{12}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{12}b_{21} & a_{22}b_{22} \end{pmatrix} \tag{D.10}$$

## D.2 Dirac's Relation

A relation that is often exploited in the book is Dirac's relation [66], which for two arbitrary vector operators $\boldsymbol{A}$ and $\boldsymbol{B}$ reads

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B}\,\mathbf{1}_2 + \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}) \tag{D.11}$$

where the (2×2) unit matrix $\mathbf{1}_2$ is usually omitted. This relation can be verified by evaluating the scalar products on the left hand side of the relation,

$$\begin{aligned} (\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) &= (\sigma_x A_x + \sigma_y A_y + \sigma_z A_z)(\sigma_x B_x + \sigma_y B_y + \sigma_z B_z) \\ &= \sigma_x^2 A_x B_x + \sigma_x\sigma_y A_x B_y + \sigma_x\sigma_z A_x B_z + \sigma_y\sigma_x A_y B_x + \sigma_y^2 A_y B_y \\ &\quad +\sigma_y\sigma_z A_y B_z + \sigma_z\sigma_x A_z B_x + \sigma_z\sigma_y A_z B_y + \sigma_z^2 A_z B_z \\ &= A_x B_x + A_y B_y + A_z B_z + \mathrm{i}\sigma_z A_x B_y - \mathrm{i}\sigma_y A_x B_z \\ &\quad -\mathrm{i}\sigma_z A_y B_x + \mathrm{i}\sigma_x A_y B_z + \mathrm{i}\sigma_y A_z B_x - \mathrm{i}\sigma_x A_z B_y \\ &= \boldsymbol{A}\cdot\boldsymbol{B} + \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}) \end{aligned} \tag{D.12}$$

if we use the relations of the Pauli spin matrices given in appendix D.1. This proof can also be given in more compact form as

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) = \sum_{i,j=1}^{3}\sigma_i A_i \sigma_j B_j \tag{D.13}$$

$$\stackrel{(D.4)}{=} \sum_{i,j=1}^{3}\left(\delta_{ij}\mathbf{1}_2 + \mathrm{i}\sum_{k=1}^{3}\varepsilon_{ijk}\sigma_k\right)A_i B_j \tag{D.14}$$

$$\overset{(A.6)}{=} \sum_{i=1}^{3} A_i B_i + \mathrm{i} \sum_{k=1}^{3} \sigma_k (\boldsymbol{A} \times \boldsymbol{B})_k \tag{D.15}$$

$$= \boldsymbol{A} \cdot \boldsymbol{B} + \mathrm{i}\boldsymbol{\sigma} \cdot (\boldsymbol{A} \times \boldsymbol{B}) \tag{D.16}$$

Obviously, if $\boldsymbol{A} = \boldsymbol{B}$ the Dirac relation simplifies to

$$(\boldsymbol{\sigma} \cdot \boldsymbol{A})^2 = \boldsymbol{A}^2 + \mathrm{i}\boldsymbol{\sigma} \cdot (\boldsymbol{A} \times \boldsymbol{A}) \tag{D.17}$$

which reads in the case of $\boldsymbol{A} = \boldsymbol{p}$

$$(\boldsymbol{\sigma} \cdot \boldsymbol{p})^2 = \boldsymbol{p}^2 \tag{D.18}$$

because $\boldsymbol{p} \times \boldsymbol{p} = 0$.

### D.2.1 Momenta and Vector Fields

The situation is more complicated in the presence of vector potentials, where we have

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) = \boldsymbol{\pi}^2 + \mathrm{i}\boldsymbol{\sigma} \cdot (\boldsymbol{\pi} \times \boldsymbol{\pi}) \tag{D.19}$$

By noting that $\boldsymbol{\pi} = \boldsymbol{p} - \frac{q_e}{c}\boldsymbol{A}$ the vector product of the kinematical momentum operator with itself can be simplified to

$$\begin{aligned} \boldsymbol{\pi} \times \boldsymbol{\pi} &= -\frac{q_e}{c} \left( \boldsymbol{p} \times \boldsymbol{A} + \boldsymbol{A} \times \boldsymbol{p} \right) \\ &= -\mathrm{i}\hbar \frac{q_e}{c} \left( \nabla \times \boldsymbol{A} + \boldsymbol{A} \times \nabla \right) \end{aligned} \tag{D.20}$$

since the vector products of the canonical momentum $\boldsymbol{p}$ and the vector potential $\boldsymbol{A}$ with themselves vanish, respectively. We now consider the action of the $i$-th component of the operator $\nabla \times \boldsymbol{A}$ on a two-component spinor $\psi^{\mathrm{L}}$,

$$\begin{aligned} \left( \nabla \times \boldsymbol{A}\psi^{\mathrm{L}} \right)_i &= \sum_{j,k=1}^{3} \varepsilon_{ijk} \partial_j (A_k \psi^{\mathrm{L}}) \\ &= \sum_{j,k=1}^{3} \varepsilon_{ijk} (\partial_j A_k) \psi^{\mathrm{L}} + \sum_{j,k=1}^{3} \varepsilon_{ijk} A_k (\partial_j \psi^{\mathrm{L}}) \\ &= B_i \psi^{\mathrm{L}} - \left( \boldsymbol{A} \times \nabla \psi^{\mathrm{L}} \right)_i \end{aligned} \tag{D.21}$$

where we have employed $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ in the last step (where now $\boldsymbol{A}$ is *not* some general vector but the electromagnetic vector potential, of course.) This immediately implies for the vector product of the kinematical momentum with itself

$$\boldsymbol{\pi} \times \boldsymbol{\pi} = \frac{\mathrm{i}\hbar q_e}{c} \boldsymbol{B} \tag{D.22}$$

and yields thus the final result

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) = \boldsymbol{\pi}^2 - \frac{q_e \hbar}{c} \boldsymbol{\sigma} \cdot \boldsymbol{B} \tag{D.23}$$

### D.2.2 Four-Dimensional Generalization

Dirac's relation can also be generalized to the four-component framework if $\boldsymbol{\sigma}$ is substituted by $\boldsymbol{\alpha}$,

$$(\boldsymbol{\alpha} \cdot \boldsymbol{A})(\boldsymbol{\alpha} \cdot \boldsymbol{B}) = \mathbf{1}_2 \otimes [(\boldsymbol{A} \cdot \boldsymbol{B})\, \mathbf{1}_2 + \mathrm{i}\boldsymbol{\sigma} \cdot (\boldsymbol{A} \times \boldsymbol{B})] \tag{D.24}$$

$$= (\boldsymbol{A} \cdot \boldsymbol{B})\, \mathbf{1}_4 + \mathrm{i}\, \boldsymbol{\Sigma} \cdot (\boldsymbol{A} \times \boldsymbol{B}) \tag{D.25}$$

because

$$(\boldsymbol{\alpha} \cdot \boldsymbol{A})(\boldsymbol{\alpha} \cdot \boldsymbol{B}) = \begin{pmatrix} (\boldsymbol{\sigma} \cdot \boldsymbol{A})(\boldsymbol{\sigma} \cdot \boldsymbol{B}) & \mathbf{0} \\ \mathbf{0} & (\boldsymbol{\sigma} \cdot \boldsymbol{A})(\boldsymbol{\sigma} \cdot \boldsymbol{B}) \end{pmatrix} \tag{D.26}$$

which is a (4×4)-matrix.