Quantum Mechanics

1 General Definitions and Equations

- Hermitian: $M = M^\dagger$, represent observables
- Anti-Hermitian/skew-Hermitian: $M = -M^\dagger$
- Unitary: $M = M^{-1}$
- Orthonormality: $\int \psi_m^*(x)\psi_n(x)dx = \delta_{nm}$
- Normalization: $\int |\psi|^2 dx = 1$, ψ lives in Hilbert space
- Eigenstate ψ, eigenvalue a of operator $\hat{A}$: $\hat{A}\psi = a\psi$
  - Eigenstates are degenerate when they have the same eigenvalue
  - Eigenstates are orthogonal for non-degenerate eigenvalues
  - Measurement of observable gives eigenvalue
  - Can have simultaneous eigenstates of two operators if operators commute
- Boundary conditions on wavefunction: $\begin{cases} \psi & \text{continuous} \\ \dfrac{d\psi}{dx} & \text{continuous except when potential is infinite} \end{cases}$
- Probability: $|\Psi(x,t)|^2\, dx = \begin{Bmatrix} \text{probability of finding particle} \\ \text{between } x \text{ and } (x+dx)\text{, at time t} \end{Bmatrix}$
  - To find most probable value, maximize probability
- Schrödinger equation: $i\hbar\dfrac{\partial\Psi}{\partial t} = H\Psi$
- Time independent Schrödinger equation: $H\Psi = E\Psi$
- Time dependence of expectation value: $\dfrac{d\langle Q\rangle}{dt} = \dfrac{i}{\hbar}\langle[H,\ Q]\rangle + \left\langle\dfrac{\partial Q}{\partial t}\right\rangle$
- Generalized uncertainty principle: $\sigma_A\sigma_B \geq \left|\dfrac{1}{2i}\langle[A,\ B]\rangle\right|^2$
- Heisenberg uncertainty principle: $\sigma_x\sigma_p \geq \dfrac{\hbar}{2}$

- Canonical commutator: $[x,\ p] = i\hbar, \quad \hat{p} = -i\hbar\frac{\partial}{\partial x}$
- Variance: $\sigma_j^2 = \left\langle (\Delta j)^2 \right\rangle = \left\langle j^2 \right\rangle - \langle j \rangle^2$
- Expectation value of operator $\hat{A} = \langle \psi \mid \hat{A} \mid \psi \rangle$
- Energy of a photon: $E = h\nu$
- deBroglie wavelength of particle: $\lambda = \frac{h}{p}$
- Fermions: half integer spin, antisymmetric wave function
- Bosons: integer spin, symmetric wave function
- Commutator relation: $[AB, C] = A[B, C] + [A, C]B$

2 Infinite Square Well, Length L

- $V = \begin{cases} 0 & \text{if } 0 \leq x \leq L \\ \infty & otherwise \end{cases}$
- $\psi_n(x) = \sqrt{\frac{2}{L}} sin\left(\frac{n\pi x}{L}\right)$
- $E = \frac{n^2\pi^2\hbar^2}{2mL^2}$
- For non-infinite square well, sinusoidal function in center of well, exponential decay outside of well

3 The Harmonic Oscillator

- $V(x) = \frac{1}{2}m\omega^2x^2$
- $\psi_0(x) = A_0 e^{-\frac{m\omega}{2\hbar}x^2} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$
- $\psi_n(x) = A_n \left(a^\dagger\right) e^{-\frac{m\omega}{2\hbar}x^2} = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$
  - where H_n are the Hermite polynomials

- $E_n = \left(n + \frac{1}{2}\right) \hbar\omega$

- $a^\dagger \psi_n = \sqrt{n+1}\ \psi_{n+1} \qquad a\psi_n = \sqrt{n}\ \psi_{n-1}$

- $\psi_n = \dfrac{1}{\sqrt{n!}} \left(a^\dagger\right)^n \psi_0$

- $a = \sqrt{\dfrac{m\omega}{2\hbar}} \left(\hat{x} + \dfrac{i}{m\omega}\hat{p}\right), \qquad a^\dagger = \sqrt{\dfrac{m\omega}{2\hbar}} \left(\hat{x} - \dfrac{i}{m\omega}\hat{p}\right)$

- $\hat{x} = \sqrt{\dfrac{\hbar}{2m\omega}} \left(a + a^\dagger\right), \qquad \hat{p} = i\sqrt{\dfrac{\hbar m\omega}{2}} \left(a^\dagger - a\right)$

4 The Free Particle

- $V(x) = 0$ *everywhere*

- Wave packet: $\Psi(x,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \phi(k) e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} dk$

- $\phi(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx$

- $k \equiv \pm\dfrac{\sqrt{2mE}}{\hbar}, \qquad \mathit{with} \begin{cases} k > 0 & \Rightarrow \text{ traveling to the right} \\ k < 0 & \Rightarrow \text{ traveling to the left} \end{cases}$

- Group velocity: $v_{group} = \dfrac{d\omega}{dk}$

- Phase velocity: $v_{phase} = \dfrac{\omega}{k}$

5 The Hydrogen Atom

- $E_n = \dfrac{-13.6\ eV}{n^2}$

- $\psi_{100}(r,\theta,\phi) = \dfrac{1}{\sqrt{\pi a^3}} e^{-r/a}$, $a = 0.53 \times 10^{-10} m$

- Selection rules for transitions:
  - $\Delta l = \pm 1, \qquad \Delta m = \pm 1, 0$

6 Angular Momentum

- Simultaneous eigenstates of L^2 and one component of $\vec{L}$ (usually L_z) are spherical harmonics
  - $Y_l^m(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|!)}{(l+|m|!)}} e^{im\phi} P_l^m(\cos\theta)$
- $L^2 \; Y_l^m(\theta,\phi) = \hbar^2 l(l+1) \; Y_l^m(\theta,\phi)$
- $L_z \; Y_l^m(\theta,\phi) = \hbar m \; Y_l^m(\theta,\phi), \quad L_z = -i\hbar\frac{\partial}{\partial\phi}$
- Commutators: $[L_x,\ L_y] = i\hbar L_z, \quad [L_y,\ L_z] = i\hbar L_x, \quad [L_z,\ L_x] = i\hbar L_y$
- Raising and lowering operators: $L_\pm = L_x \pm iL_y$
- $L_\pm \; Y_l^m(\theta,\phi) = \hbar\sqrt{l(l+1) - m(m\pm 1)} \; Y_l^{m\pm 1}(\theta,\phi)$

7 Spin 1/2

- Pauli Matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_i = \frac{\hbar}{2}\sigma_i$
- Addition of two spin 1/2 particles: symmetric triplet and anti-symmetric singlet
  - $s = 1 \Rightarrow \begin{cases} |1\ 1\rangle = |\uparrow\uparrow\rangle \\ |1\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1\ \text{-}1\rangle = |\downarrow\downarrow\rangle \end{cases}$
  - $s = 0 \Rightarrow \begin{cases} |0\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \end{cases}$

8 Time-independent Perturbation Theory

- First order energy correction: $E_n^1 = \langle \psi_n^0 \mid H' \mid \psi_n^0 \rangle$
- First order wave function correction: $\psi_n^1 = \sum_{m \neq n} \frac{\langle \psi_m^0 \mid H' \mid \psi_n^0 \rangle}{(E_n^0 - E_m^0)} \psi_m^0$
- Second order energy correction: $E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_m^0 \mid H' \mid \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$